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Winding strings in AdS_3

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ABSTRACT: Correlation functions of one-unit spectral flowed states in string theory on AdS_3 are considered. We present the modified Knizhnik-Zamolodchikov and null vector equations to be satisfied by amplitudes containing states in winding sector one and study their solution corresponding to the four point function including one $w = 1$ field. We compute the three point function involving two one-unit spectral flowed operators and find expressions for amplitudes of three $w = 1$ states satisfying certain particular relations among the spins of the fields. Several consistency checks are performed.

KEYWORDS: Bosonic Strings, Conformal Field Models in String Theory.

Contents

1. Introduction

Correlation functions of operators creating string states in three dimensional anti-de Sitter space (AdS_3) are essential ingredients to establish the consistency of string theory in this geometry as well as to explore the AdS/CFT correspondence beyond the supergravity approximation. However the non-rational structure of the SL(2,$\mathbb{R}$) CFT describing the worldsheet of strings propagating in AdS_3 presents some difficulties to the computation of these correlators, since very little is known about non-compact conformal field theories in general.

Nevertheless, amplitudes of some physical states were computed in reference [1]. The starting point in these calculations is the SL(2,$\mathbb{C}$)/SU(2) WZW model which describes the worldsheet of strings propagating in the hyperbolic space H_3. Two, three and four point functions of primary fields in this coset model were derived in references [2, 3]. The connection between these correlators and some amplitudes in the Lorentzian theory described by the SL(2,$\mathbb{R}$) WZW model was performed in [1] using the equivalence between string theory on AdS_3 and the dual two dimensional CFT on the boundary. Actually, the

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interpretation of these amplitudes as correlation functions of the dual CFT is crucial to determine the correlators and to establish the structure of the factorization of four point functions. In this way the closure of the operator algebra on the Hilbert space of string theory on AdS_3 [4, 5] was verified in [1] for four point functions of primary fields related to primaries of the $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$ coset model through analytic continuation.

However not all the states in the physical spectrum of string theory on AdS_3 can be obtained analytically continuing fields in H_3. Actually, the spectral flow symmetry of the $\mathrm{SL}(2,\mathbb{R})$ WZW model establishes the occurrence of winding states created by spectral flowed operators [4]. An important auxiliary tool to construct these states is the spectral flow operator which allows to interpolate between objects in different winding sectors w. Two and certain three point functions containing spectral flowed fields were computed in [1] and used to verify both the factorization of four point functions of $w = 0$ fields into products of three point functions summed over intermediate physical states and the pattern of spectral flow number conservation of the correlators determined by the $\mathrm{SL}(2,\mathbb{R})$ current algebra.

Two alternative methods were proposed in [1] to evaluate N-point functions containing states in winding sectors $w \neq 0$. Both procedures involve the insertion of one auxiliary spectral flow operator for each winding unit. This implies the calculation of expectation values of more than N vertex operators. The two approaches were applied to compute two point functions in arbitrary winding sectors and three point functions involving one $w = 1$ and two $w = 0$ states, but more general three and four point functions require the calculation of correlators with five or more operator insertions, with the consequent complications. These general amplitudes are needed to definitely settle the question about the unitarity of the theory. Indeed the structure of the factorization of four point functions should be consistent with the physical Hilbert space of string theory, but four point functions involving spectral flowed states have not been computed so far. In this paper we give one step forward towards this project by computing the three point function containing two $w = 1$ states from the five point function involving two spectral flow operators. We also study the four point function including one $w = 1$ state starting from the five point function containing only one spectral flow operator. Besides we find expressions for amplitudes of three $w = 1$ states satisfying certain particular relations among the spins of the fields.

As is well known expectation values of fields in WZW models must obey the Knizhnik-Zamolodchikov (KZ) equations [6], a system of linear differential equations which follow from the Sugawara construction of the energy-momentum tensor. An important additional property of WZW models for compact groups is the existence of null vectors in the Verma modules of the primary fields. These give additional differential equations which allow to determine the fusion rules and eventually solve the theory [7]. Unfortunately the unitary representations of $\mathrm{SL}(2,\mathbb{R})$ which give rise to the physical spectrum of string theory on AdS_3 do not contain singular vectors. Nevertheless the spectral flow operator has a null current algebra descendant which plays a relevant role in this non-rational theory. Indeed it adds one differential equation for each unit of spectral flow of the operators involved in the amplitudes. Moreover this null state allows to simplify the KZ equations in the coordinates labeling the position of the spectral flow operators [1].

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Despite all this information the differential equations to be obeyed by correlation functions containing spectral flowed operators are difficult to solve because, as we will see, they turn out to give iterative relations. Actually we shall show in the following sections that the KZ and null vector equations for amplitudes of states in winding sectors $w \neq 0$ relate two or more expectation values in which the spectral flowed fields at a given position have different spins and conformal dimensions. Nonetheless we shall manipulate and solve the system of iterative equations in certain particular cases needed to obtain correlators of three $w = 1$ string states and four point functions involving one $w = 1$ state.

The organization of this paper is as follows. For completeness and in order to introduce our notations and conventions, in the next section we briefly review the spectrum of string theory on AdS_3, recall the construction of one-unit spectral flowed operators and collect the results of the amplitudes involving such fields which have been obtained so far. In section 3 we compute the three point function involving two $w = 1$ fields and study its pole structure. As a consistency check, we verify that it correctly reproduces the two point function of a $w = 1$ field when the third operator is the identity. In addition we find, as a byproduct, the four point function including one $w = 1$ field along with a spectral flow operator. In section 4, we discuss the Ward identities to be satisfied by correlation functions containing $w = 1$ fields and we deduce the modified KZ and null vector equations that they must obey. The solution to these equations is analyzed in section 5 for the case of the four point function of one $w = 1$ field and three unflowed generic states. This is done by first expanding the correlator in powers of the corresponding cross ratio coordinate. We explicitly find the lowest order contribution and write an iterative differential equation for the higher orders in terms of the lowest one. Two consistency checks are succesfully performed. First we verify that, for appropriate choices of the spins, the correlator properly reduces to the three point function involving one $w = 1$ field, as computed in [1]. Then we also verify that the functional form of the four point function including one $w = 1$ field and a spectral flow operator computed in section 3 is correctly reproduced. Conclusions and discussions are offered in section 6.

We have included three appendices. Some properties of five and six point functions containing states with generic spin along with spectral flow operators are listed in appendix A, whereas appendix B presents some useful formulae which are used in the main body of the article. In appendix C we compute expressions of the three point function involving three $w = 1$ operators for certain particular relations among the spins of the fields. This is done by proposing an *ansatz* for the solution of the modified KZ and null vector equations.

2. Perturbative string theory on AdS_3

In this section we gather known results about the spectrum and correlation functions of perturbative string theory on AdS_3 in order to set up our conventions. We follow the same notation as reference [1] and so the expert reader can proceed directly to the next section.

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2.1 Notation and conventions

The Hilbert space of the WZW model is a sum of products of representations of the SL(2,$\mathbb{R}$) current algebra given by

$$
\begin{aligned}
[J^3_n, J^3_m] &= -\frac{k}{2}\, n\delta_{n+m,0} \ , \\
[J^3_n, J^\pm_m] &= \pm J^\pm_{n+m} \ , \\
[J^+_n, J^-_m] &= -2J^3_{n+m} + kn\delta_{n+m,0} \ ,
\end{aligned}
$$

and the same for $\bar{J}^{3,\pm}$. The Sugawara construction of the energy-momentum tensor

$$
T(z) = \frac{1}{k-2}\left(J^+(z)J^-(z) - J^3(z)J^3(z)\right) \ ,
$$

determines the Virasoro generators

$$
\begin{aligned}
L_0 &= \frac{1}{k-2}\left[\frac{1}{2}(J^+_0 J^-_0 + J^-_0 J^+_0) - (J^3_0)^2 + \sum_{m=1}^{\infty}(J^+_{-m}J^-_m + J^-_{-m}J^+_m - 2J^3_{-m}J^3_m)\right] \ , \\
L_{n\neq 0} &= \frac{1}{k-2}\sum_{m=1}^{\infty}(J^+_{n-m}J^-_m + J^-_{n-m}J^+_m - 2J^3_{n-m}J^3_m) \ ,
\end{aligned}
$$

which obey the following commutation relations

$$
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \ .
$$

The central charge is given by

$$
c = \frac{3k}{k-2} \ ,
$$

with $k > 2$.

The physical states of string theory on AdS$_3$ are in unitary representations of the universal cover of SL(2,$\mathbb{R}$) [4]. These are generated from the continuous ($\mathcal{C}^\alpha_j$) and the lowest ($\mathcal{D}^+_j$) and highest ($\mathcal{D}^-_j$) weight discrete representations of the zero modes acting with $J^{3,\pm}_{-n}, n > 0$. In our conventions these are

$$
\begin{aligned}
\mathcal{D}^+_j &= \{|j;m\rangle \ : \ m = j,\ j+1,\ j+2, \ldots\} \ , \\
\mathcal{D}^-_j &= \{|j;m\rangle \ : \ m = -j,\ -j-1,\ -j-2, \ldots\} \ ,
\end{aligned}
$$

with

$$
\frac{1}{2} < j < \frac{k-1}{2} \ , \tag{2.1}
$$

and

$$
\mathcal{C}^\alpha_j = \{|j,\alpha;m\rangle \ : \ m = \alpha,\ \alpha \pm 1,\ \alpha \pm 2, \ldots, \alpha \in R\} \ , \qquad j = \frac{1}{2} + is \ , \quad s \in \mathbb{R} \ .
$$

The conformal weight of the primary fields is given by

$$
\Delta_j = -\frac{j(j-1)}{k-2} \ , \tag{2.2}
$$

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and similarly for $\bar{\Delta}_{\bar{j}}$. The current algebra descendants of the primary operators contribute an additional integer for each excitation level. In the string theory application one can consider the spacetime to be a product of AdS_3 times an internal manifold. In this case the conformal weight of the physical states may be supplemented with a contribution from the internal CFT, usually denoted h. Moreover the physical state conditions for string states, namely $L_0|\Psi\rangle = |\Psi\rangle, L_n|\Psi\rangle = 0, n > 0$ and $L_0 = \bar{L}_0$ determine $j = \bar{j}$.

The spectral flow automorphism

$$\tilde{J}^3_n = J^3_n - \frac{k}{2}\, w\delta_{n,0}\ , \qquad \tilde{J}^\pm_n = J^\pm_{n\pm w}\ , \qquad \tilde{L}_n = L_n + wJ^3_n - \frac{k}{4}\, w^2\delta_{n,0}\ ,$$

parametrized with $w \in \mathbf{Z}$, generates new representations defined by

$$\begin{aligned} &J^\pm_{n\pm w}|j,m,w\rangle = 0\ , \qquad J^3_n|j,m,w\rangle = 0\ , \qquad (n \geq 1)\ , \\ &J^3_0|j,m,w\rangle = \left(m + \frac{k}{2}\, w\right)|j,m,w\rangle\ . \end{aligned}$$

The conformal weight of the primary fields (2.2) transforms consequently as

$$\Delta^w_j = -\frac{j(j-1)}{k-2} - mw - \frac{k}{4}\, w^2\ , \tag{2.3}$$

and similarly for $\bar{\Delta}^{\bar{w}}_j$. Periodicity of the closed string under the worldsheet coordinate transformation $\sigma \to \sigma + 2\pi$ settles $w = \bar{w}$. These spectral flowed states have to be added to the unflowed fields of the full representations generated from $\mathcal{D}^\pm_j$ and $\mathcal{C}^\alpha_j$ in order to describe the complete spectrum of the theory.

The primary states in the sector $w = 0$ can be represented by an operator $\Phi_j(x,\bar{x};w,\bar{w})$ which satisfies the following OPE with the currents

$$J^a(z)\Phi_j(x,\bar{x};w,\bar{w}) \sim \frac{D^a}{z-w}\, \Phi_j(x,\bar{x};w,\bar{w})\ , \qquad (a = 3,\pm)\ ,$$

where the differential operators

$$D^+ = \frac{\partial}{\partial x}\ , \qquad D^3 = x\frac{\partial}{\partial x} + j\ , \qquad D^- = x^2\frac{\partial}{\partial x} + 2jx\ ,$$

give a representation of the Lie algebra of SL(2). Here $x, \bar{x}$ keep track of the SL(2) weights of the fields and they are interpreted as the coordinates of the boundary in the AdS/CFT context.

One can also consider operators in the m basis, obtained through the following transformation from the x basis

$$\Phi_{j;m,\bar{m}} = \int \frac{d^2x}{|x|^2} x^{j-m}\bar{x}^{j-\bar{m}}\Phi_j(x,\bar{x})\ , \tag{2.4}$$

where $m - \bar{m}$ is an integer.

In the sector $w = 1$ the spectral flowed states are constructed by the fusion of Φ_j with the spectral flow operator $\Phi_{\frac{k}{2}}$ through the following operation [1]

$$\begin{aligned} \Phi^{w=1,j}_{J,\bar{J}}(x,\bar{x};z,\bar{z}) \equiv \lim_{\epsilon,\bar{\epsilon}\to 0} \epsilon^m\bar{\epsilon}^{\bar{m}} \int d^2y\, y^{j-m-1}\bar{y}^{j-\bar{m}-1} \\ \times\Phi_j(x+y,\bar{x}+\bar{y};z+\epsilon,\bar{z}+\bar{\epsilon})\Phi_{\frac{k}{2}}(x,\bar{x};z,\bar{z})\ , \end{aligned} \tag{2.5}$$

where

$$J = m + \frac{k}{2} \ , \qquad\qquad \bar{J} = \bar{m} + \frac{k}{2} \ , \tag{2.6}$$

denote the left and right spins of the $w = 1$ field. In the x basis, the winding number w turns out to be always positive, unlike in the m basis where the sign of w is correlated with the sign of m, thus distinguishing by convention incoming from outgoing spectral flowed states in the correlation functions.

Alternatively $\Phi^{w=1,j}_{J,\bar{J}}$ can be also defined by integrating over ϵ as

$$\begin{aligned}\Phi^{w=1,j}_{J,\bar{J}}(x,\bar{x};z,\bar{z}) \equiv \lim_{y,\bar{y}\to 0} y^{j-m}\bar{y}^{j-\bar{m}} \int d^2\epsilon \ \epsilon^{m-1}\bar{\epsilon}^{\bar{m}-1} \times \\ \times \Phi_j(x+y,\bar{x}+\bar{y};z+\epsilon,\bar{z}+\bar{\epsilon})\Phi_{\frac{k}{2}}(x,\bar{x};z,\bar{z}) \ .\end{aligned} \tag{2.7}$$

Both definitions (2.5) and (2.7) are equivalent and they are understood to hold inside correlation functions. The first one is local in $z, \bar{z}$ whereas the second one is local in $x, \bar{x}$. As pointed out in [1], the limit $\epsilon \to 0$ in (2.5) – and similarly $y \to 0$ in (2.7) – exists and it verifies several important checks. For instance it has the right operator product expansions with the currents, namely

$$\begin{aligned}J(x',z')\Phi^{w=1,j}_{J,\bar{J}}(x,z) &= -(j-m-1)\frac{(x-x')^2}{(z'-z)^2}\,\Phi^{w=1,j}_{J+1,\bar{J}}(x,z) \\ &+\frac{1}{z'-z}\left[(x-x')^2\frac{\partial}{\partial x}+2\left(m+\frac{k}{2}\right)(x-x')\right]\Phi^{w=1,j}_{J,\bar{J}}(x,z) \ ,\end{aligned} \tag{2.8}$$

where

$$J(x,z) = -J^-(z) + 2xJ^3(z) - x^2J^+(z) \ .$$

Furthermore the definition (2.5) reproduces the expressions derived in [1] for the two point function of spectral flowed states and the three point function which includes in addition two other operators in the sector $w = 0$.

Vertex operators for string states in higher winding sectors can be easily obtained in the m basis where they are expressed in terms of SL(2) parafermions and one free boson [4] (see [8] for the free field representation). However, as the winding number increases, they become more complicated in the x basis.

2.2 Correlation functions of winding states

Correlation functions of unflowed states were computed in [1] performing analytic continuation on the results for the Euclidean SL(2,$\mathbb{C}$)/SU(2) WZW model obtained in [2, 3]. As discussed in [1], correlators including spectral flowed states can be evaluated in the m basis starting from expectation values of states in the $w = 0$ sector including spectral flow operators. Alternatively one can perform the spectral flow operation directly in the x basis using the definition (2.5).

The two point function of spectral flowed states was computed in [1] (see also [8] for a derivation in the m basis using the free field theory approach) and it is the following

$$\langle \Phi^{w,j}_{J,\bar{J}}(x_1,z_1)\Phi^{w,j'}_{J,\bar{J}}(x_2,z_2)\rangle = x_{12}^{-2J}\bar{x}_{12}^{-2\bar{J}}z_{12}^{-2\Delta^w_j}\bar{z}_{12}^{-2\bar{\Delta}^w_j} \tag{2.9}$$
$$\times\left[\delta(j+j'-1)+\delta(j-j')\frac{\pi B(j)}{\gamma(2j)}\frac{\Gamma(j+m)}{\Gamma(1-j+m)}\frac{\Gamma(j-\bar{m})}{\Gamma(1-j-\bar{m})}\right],$$

where Δ^w_j is given in (2.3) and

$$B(j)=\frac{k-2}{\pi}\frac{\nu^{1-2j}}{\gamma\left(\frac{2j-1}{k-2}\right)}\ ,\quad \nu=\pi\,\frac{\Gamma\left(\frac{k-3}{k-2}\right)}{\Gamma\left(\frac{k-1}{k-2}\right)}\ , \tag{2.10}$$

$$\gamma(x)\equiv\frac{\Gamma(x)}{\Gamma(1-x)}\ . \tag{2.11}$$

Recall that in the x basis the operators are labeled with positive w, so this two point function conserves winding number as expected. As is well known an N point function can violate winding number by $N-2$ units [1, 9].[1] The three point function including one operator in the $w=1$ sector is the following[2]

$$\begin{aligned}&\left\langle \Phi^{w=1,j_1}_{J,\bar{J}}(x_1,z_1)\Phi_{j_2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\right\rangle =\\ &\qquad = B(j_1)C\left(\frac{k}{2}-j_1,j_2,j_3\right)\pi\,\frac{1}{\gamma(j_1+j_2+j_3-k/2)}\\ &\qquad\times\frac{\Gamma(j_1+J-\frac{k}{2})}{\Gamma(1+J-j_2-j_3)}\,\frac{\Gamma(j_2+j_3-\bar{J})}{\Gamma(1-j_1-\bar{J}+\frac{k}{2})}\\ &\qquad\times\left(x_{21}^{j_3-j_2-J}x_{31}^{j_2-j_3-J}x_{32}^{J-j_2-j_3}\right)\left(z_{21}^{\Delta_3-\Delta_2-\Delta_1^{w=1}}z_{31}^{\Delta_2-\Delta_3-\Delta_1^{w=1}}z_{32}^{\Delta_1^{w=1}-\Delta_2-\Delta_3}\right)\\ &\qquad\times(\text{antiholomorphic part})\ ,\end{aligned} \tag{2.12}$$

where J is given in (2.6) and $\Delta_1^{w=1}$ is (see (2.3))

$$\Delta_1^{w=1}=\Delta_1-J+\frac{k}{4}\ . \tag{2.13}$$

$B(j)$ is given in (2.10) and $C(j_1,j_2,j_3)$ is the coefficient corresponding to the amplitude of three $w=0$ fields, namely

$$C(j_1,j_2,j_3)=-\frac{G(1-j_1-j_2-j_3)G(j_3-j_1-j_2)G(j_2-j_3-j_1)G(j_1-j_2-j_3)}{2\pi^2\nu^{j_1+j_2+j_3-1}\gamma\left(\frac{k-1}{k-2}\right)G(-1)G(1-2j_1)G(1-2j_2)G(1-2j_3)}\ , \tag{2.14}$$

where

$$G(j)=(k-2)^{\frac{j(k-1-j)}{2(k-2)}}\Gamma_2(-j\mid 1,k-2)\Gamma_2(k-1+j\mid 1,k-2)\ ,$$

[1] See appendix D of [1] for a detailed analysis.

[2] Actually this expression differs from the one in [1] by an irrelevant factor $(-1)^{J-\bar{J}}$, as it can be verified using the property $J-\bar{J}\in\mathbf{Z}$ together with the identity $\Gamma(x)\Gamma(1-x)=\frac{\pi}{sin(\pi x)}$.

and $\Gamma_2(x|1,\omega)$ is the Barnes double Gamma function which reads

$$\log(\Gamma_2(x \mid 1,\omega)) = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}\left[\sum_{n,m=0}^{\infty}(x+n+m\omega)^{-\epsilon} - \sum_{n,m=0\ ;\ (n,m)\neq(0,0)}^{\infty}(n+m\omega)^{-\epsilon}\right] .$$

This three point function was obtained in [1] by first computing a four point function including one spectral flow operator $\Phi_{\frac{k}{2}}$. Such calculation is performed by explicitly solving the corresponding KZ and null vector equations. The four point function gives rise to (2.12) after spectral flowing as in the definition (2.5) or alternatively, after transforming to the m basis, extracting the pole residue at $m=-\frac{k}{2}$ and acting with the spectral flow operator on the unflowed field Φ_{j_1}.

This summarizes the already known explicit expressions for correlators including spectral flowed fields. Note that, whereas the two point function is known for fields in unlimited winding sectors, the situation gets more complicated in the case of the three point function, where only the case involving one $w=1$ and two $w=0$ operators has been computed so far. The increasing difficulties to compute three point functions including additional spectral flowed fields are due to the fact that one has to start from amplitudes containing more spectral flow operators. In the following section we shall illustrate this statement by going one step further and computing the amplitude of two $w=1$ and one $w=0$ states in the x basis starting from the five point function which includes two spectral flow operators.

3. Three point function involving two $w=1$ fields

We want to compute the following three point function including two $w=1$ fields

$$\left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi_{j_2}(x_2,z_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\right\rangle . \tag{3.1}$$

The starting point is the five point function with two spectral flow operators, namely

$$A_5 \equiv \langle \Phi_{\frac{k}{2}}(x_1,z_1)\Phi_{\frac{k}{2}}(x_2,z_2)\Phi_{j_1}(y_1,\zeta_1)\Phi_{j_2}(y_2,\zeta_2)\Phi_{j_3}(y_3,\zeta_3)\rangle . \tag{3.2}$$

Due to the spectral flow operators inserted at (x_1,z_1) and (x_2,z_2), A_5 must obey the null vector equations

$$0 = \sum_{i=2}^{5}\frac{x_i-x_1}{z_1-z_i}\left[(x_i-x_1)\frac{\partial}{\partial x_i}+2j_i\right] A_5 , \tag{3.3}$$

at (x_1,z_1) and a similar one at (x_2,z_2), where we have renamed the insertion points $y_i = x_{i+2}, \zeta_i = z_{i+2}$. Moreover the singular state condition allows to simplify the action of the Sugawara construction on $\Phi_{\frac{k}{2}}$ as

$$L_{-1}|j=k/2\rangle = -J^3_{-1}|j=k/2\rangle. \tag{3.4}$$

Therefore the insertion of the spectral flow operators implies the following reduced form of the corresponding KZ equation

$$\frac{\partial A_5}{\partial z_1} = -\sum_{i=2}^{5}\frac{1}{z_1-z_i}\left[(x_i-x_1)\frac{\partial}{\partial x_i}+j_i\right] A_5 , \tag{3.5}$$

and similarly for (x_2,z_2).

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The x_i dependence of the solution to the null vector equations was found in [9] (see also [1]). Here we give the complete solution including the dependence on the worldsheet coordinates z_i, which is determined from the Ward identities and the reduced KZ equations (3.5), namely

$$\begin{aligned} A_5 = {} & B(j_1)B(j_3)C\left(\frac{k}{2}-j_1, j_2, \frac{k}{2}-j_3\right)|z_{12}|^k|z_{13}|^{-2j_1}|z_{14}|^{-2j_2}|z_{15}|^{-2j_3} \\ & \times|z_{23}|^{-2j_1}|z_{24}|^{-2j_2}|z_{25}|^{-2j_3}|z_{34}|^{2(\Delta_3-\Delta_1-\Delta_2)}|z_{35}|^{2(\Delta_2-\Delta_1-\Delta_3)}|z_{45}|^{2(\Delta_1-\Delta_2-\Delta_3)} \\ & \times|x_{12}|^{2(j_1+j_2+j_3-k)}|\mu_1|^{2(j_1-j_2-j_3)}|\mu_2|^{2(j_2-j_1-j_3)}|\mu_3|^{2(j_3-j_1-j_2)}\,, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned} \mu_1 &= \frac{x_{14}x_{25}}{z_{14}z_{25}} - \frac{x_{15}x_{24}}{z_{15}z_{24}}\,, \\ \mu_2 &= \frac{x_{15}x_{23}}{z_{15}z_{23}} - \frac{x_{13}x_{25}}{z_{13}z_{25}}\,, \\ \mu_3 &= \frac{x_{13}x_{24}}{z_{13}z_{24}} - \frac{x_{14}x_{23}}{z_{14}z_{23}}\,. \end{aligned} \tag{3.7}$$

The way to determine the coefficient

$$C_5(j_1,j_2,j_3) = B(j_1)B(j_3)C\left(\frac{k}{2}-j_1, j_2, \frac{k}{2}-j_3\right)\,, \tag{3.8}$$

is reviewed in appendix A where we also list some properties of $B(j)$ and $C(j_1,j_2,j_3)$ which are useful for the calculations below.

It can be verified that this result reduces to the four point function involving one spectral flow operator computed in reference [1] when one of the generic spins vanishes. Indeed taking for instance $j_2 = 0$ in the five point function (3.6), the coefficient C_5 gives $B(j_1)\delta(j_1-j_3)$ and the x_i, z_i dependence reproduces the correlation function $\langle\Phi_{j_1}\Phi_{\frac{k}{2}}\Phi_{j_3}\Phi_{j_4}\rangle$ (with the obvious change in labels) given in equation (5.25) of reference [1] when another field has spin $\frac{k}{2}$. Actually taking $j_1 = \frac{k}{2}$ in the equation computed by J. Maldacena and H. Ooguri, the j_i dependent coefficient reduces to $B(j_3)\delta(j_3-j_4)$ and the coordinate dependence in both expressions matches, with the obvious renaming of spins and points.

As an intermediate step before computing the three point function we spectral flow once to obtain the following four point function

$$A_4^{w=1} = \langle\Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi_{\frac{k}{2}}(x_2,z_2)\Phi_{j_2}(y_2,\zeta_2)\Phi_{j_3}(y_3,\zeta_3)\rangle\,. \tag{3.9}$$

This auxiliary result will also be useful for the computation of the four point function involving one spectral flowed and three unflowed generic states which we perform in section 5. Applying the prescription (2.5) to A_5 and setting $y_1 = x_1 + t, \zeta_1 = z_1 + \epsilon$ we have to compute

$$\begin{aligned} A_4^{w=1} &= \lim_{\epsilon,\bar\epsilon\to 0}\epsilon^{m_1}\bar\epsilon^{\overline{m}_1}\int t^{j_1-m_1-1}\bar{t}^{j_1-\overline{m}_1-1}A_5(x_1,z_1,x_2,z_2,x_1+t,z_1+\epsilon,y_2,\zeta_2,y_3,\zeta_3)\;d^2t \\ &= C_5(j_1,j_2,j_3)|x_{12}|^{2(j_1+j_2+j_3-k)}|\mu_1|^{2(j_1-j_2-j_3)}\lim_{\epsilon,\bar\epsilon\to 0}\epsilon^{m_1}\bar\epsilon^{\overline{m}_1}\int d^2t\; t^{j_1-m_1-1}\bar{t}^{j_1-\overline{m}_1-1} \\ &\quad\times\left|\frac{x_{15}(x_{21}-t)}{z_{15}(z_{21}-\epsilon)} - \frac{tx_{25}}{\epsilon z_{25}}\right|^{2(j_2-j_1-j_3)}\left|\frac{tx_{24}}{\epsilon z_{24}} - \frac{(x_{21}-t)x_{14}}{(z_{21}-\epsilon)z_{14}}\right|^{2(j_3-j_1-j_2)}\,, \end{aligned}$$

where we have omitted the overall dependence on the worldsheet coordinates z_i. The integrand can be easily taken to the form $t^p\bar{t}^{\bar{p}}|at+b|^{2q}|ct+d|^{2r}$ and we can then use an identity found in reference [10] which we recall in appendix B, equation (B.1), to perform the integration. Taking next the limit $\epsilon, \overline{\epsilon} \to 0$ the $\epsilon, \overline{\epsilon}$ dependence cancels and we finally obtain

$$\begin{aligned}
A_4^{w=1} &= 2i\pi(-1)^{m_1+\bar{m}_1}B(j_1)B(j_3)C\left(\frac{k}{2}-j_1, j_2, \frac{k}{2}-j_3\right)\gamma(j_2-j_1-j_3+1) \\
&\quad \times \frac{\Gamma(j_1-J_1+\frac{k}{2})\Gamma(j_3-j_2+\bar{J}_1-\frac{k}{2})}{\Gamma(1-j_1+\bar{J}_1-\frac{k}{2})\Gamma(j_2-j_3-J_1+\frac{k}{2}+1)} \\
&\quad \times \Bigg[{}_2F_1\left(j_1+j_2-j_3, j_1-J_1+\frac{k}{2}, j_2-j_3-J_1+\frac{k}{2}+1; u\right) \\
&\qquad \times {}_2F_1\left(j_1+j_2-j_3, j_1-\bar{J}_1+\frac{k}{2}, j_2-j_3-\bar{J}_1+\frac{k}{2}+1; \bar{u}\right) \\
&\qquad + \lambda u^{j_3+J_1-j_2-\frac{k}{2}}\bar{u}^{j_3+\bar{J}_1-j_2-\frac{k}{2}}{}_2F_1\left(j_1+j_3-j_2, j_1+J_1-\frac{k}{2}, j_3+J_1-j_2-\frac{k}{2}+1; u\right) \\
&\qquad \times {}_2F_1(j_1+j_3-j_2, j_1+\bar{J}_1-\frac{k}{2}, j_3+\bar{J}_1-j_2-\frac{k}{2}+1; \bar{u})\Bigg] \\
&\quad \times x_{12}^{j_2+j_3-J_1-\frac{k}{2}}\bar{x}_{12}^{j_2+j_3-\bar{J}_1-\frac{k}{2}}|x_{14}|^{-4j_2}x_{15}^{j_2-j_3-J_1+\frac{k}{2}}\bar{x}_{15}^{j_2-j_3-\bar{J}_1+\frac{k}{2}} \\
&\quad \times x_{25}^{J_1-\frac{k}{2}-j_2-j_3}\bar{x}_{25}^{\bar{J}_1-\frac{k}{2}-j_2-j_3}z_{14}^{\Delta_3-\Delta_1^w-\Delta_2-\Delta_{k/2}}\bar{z}_{14}^{\Delta_3-\bar{\Delta}_1^w-\Delta_2-\Delta_{k/2}}z_{15}^{\Delta_2-\Delta_1^w-\Delta_3+\Delta_{k/2}} \\
&\quad \times \bar{z}_{15}^{\Delta_2-\bar{\Delta}_1^w-\Delta_3+\Delta_{k/2}}z_{45}^{\Delta_{k/2}+\Delta_1^w-\Delta_3-\Delta_2}\bar{z}_{45}^{\Delta_{k/2}+\bar{\Delta}_1^w-\Delta_3-\Delta_2}|z_{25}|^k \\
&\quad \times z^{J_1}\bar{z}^{\bar{J}_1}|1-u|^{2(j_1-j_2-j_3)}|1-z|^{-2j_2}\,.
\end{aligned} \tag{3.10}$$

Here $x=\frac{x_{12}x_{45}}{x_{14}x_{25}}$, $z=\frac{z_{12}z_{45}}{z_{14}z_{25}}$, $u=\frac{1-x}{1-z}$ and

$$\begin{aligned}
\lambda &= \frac{\gamma(j_1+j_3-j_2)\Gamma(j_2-j_3-J_1+\frac{k}{2}+1)\Gamma(j_1+\bar{J}_1-\frac{k}{2})}{\gamma(j_1+j_2-j_3)\Gamma(j_3+\bar{J}_1-j_2-\frac{k}{2}+1)\Gamma(j_3+\bar{J}_1-j_2-\frac{k}{2})} \\
&\quad \times \frac{\Gamma(\bar{J}_1-\frac{k}{2}-j_1+1)\Gamma(j_2-j_3-J_1+\frac{k}{2})}{\Gamma(j_1-J_1+\frac{k}{2})\Gamma(-j_1-J_1+\frac{k}{2}+1)}\,.
\end{aligned} \tag{3.11}$$

Now, the three point function (3.1) can be obtained either spectral flowing once more from this four point function or fusing two physical fields in the five point function (3.6), say $\Phi_{j_1}(y_1,\zeta_1)$ and $\Phi_{j_2}(y_2,\zeta_2)$, with the spectral flow operators through the prescription (2.5). The first procedure is useful to determine the coordinate dependence of the three point function whereas the second one is more convenient to obtain the spin dependent coefficient. Therefore we present both.

Let us first start from the four point function (3.10). It is convenient to rename $x_2, z_2 \to x_3, z_3$ and set $x_5 \equiv y_3 = x_3 + s$ and $z_5 \equiv \zeta_3 = z_3 + \xi$. By definition we have

$$\begin{aligned}
A_3^{w=1,w=1} &= \langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi_{j_2}(y_2,\zeta_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\rangle \\
&= C_5(j_1,j_2,j_3)\gamma(j_2-j_1-j_3+1)\frac{\Gamma(j_1-J_1+\frac{k}{2})\Gamma(j_3-j_2+\bar{J}_1-\frac{k}{2})}{\Gamma(1-j_1+\bar{J}_1-\frac{k}{2})\Gamma(j_2-j_3-J_1+\frac{k}{2}+1)}
\end{aligned}$$

$$
\begin{aligned}
&\times 2i(-1)^{m_1+\overline{m}_1}\pi x_{13}^{j_2+j_3-J_1-\frac{k}{2}}\bar{x}_{13}^{j_2+j_3-\bar{J}_1-\frac{k}{2}}|x_{14}|^{-4j_2}\\
&\times z_{14}^{\Delta_3-\Delta_1^w-\Delta_2-\Delta_{\frac{k}{2}}}\bar{z}_{14}^{\Delta_3-\bar{\Delta}_1^w-\Delta_2-\Delta_{\frac{k}{2}}}\\
&\times \lim_{\xi,\bar{\xi}\to 0}\xi^{m_2+\frac{k}{2}}\bar{\xi}^{\overline{m}_2+\frac{k}{2}}(z_{13}-\xi)^{\Delta_2-\Delta_3-\Delta_1^w+\Delta_{\frac{k}{2}}}(\bar{z}_{13}-\bar{\xi})^{\Delta_2-\Delta_3-\bar{\Delta}_1^w+\Delta_{\frac{k}{2}}}\\
&\times(z_{43}-\xi)^{-\Delta_2-\Delta_3+\Delta_1^w+\Delta_{\frac{k}{2}}}(\bar{z}_{43}-\bar{\xi})^{-\Delta_2-\Delta_3+\bar{\Delta}_1^w+\Delta_{\frac{k}{2}}}z^{J_1}\bar{z}^{\bar{J}_1}|1-z|^{-2j_2}\\
&\times\int d^2s\ s^{j_3-m_3-1}\bar{s}^{j_3-\overline{m}_3-1}s^{J_1-\frac{k}{2}-j_2-j_3}\bar{s}^{\bar{J}_1-\frac{k}{2}-j_2-j_3}\\
&\times(x_{13}-s)^{j_2-j_3-J_1+\frac{k}{2}}(\bar{x}_{13}-\bar{s})^{j_2-j_3-\bar{J}_1+\frac{k}{2}}|1-u|^{2(j_1-j_2-j_3)}\\
&\times\left[{}_2F_1\left(j_1+j_2-j_3,j_1-J_1+\frac{k}{2},j_2-j_3-J_1+\frac{k}{2}+1;u\right)\overline{{}_2F_1}(\bar{u})\right.\\
&\qquad+\lambda\ u^{j_3+J_1-j_2-\frac{k}{2}}\ \bar{u}^{j_3+\bar{J}_1-j_2-\frac{k}{2}}\\
&\qquad\left.\times{}_2F_1\left(j_1+j_3-j_2,j_1+J_1-\frac{k}{2},j_3+J_1-j_2-\frac{k}{2}+1;u\right)\overline{{}_2F_1}(\bar{u})\right],
\end{aligned}
\tag{3.12}
$$

where $\overline{{}_2F_1}$ denotes the hypergeometric function in the previous factor with the replacement $J_1\to\bar{J}_1$ in the arguments. It is convenient to change variables as suggested by the following definition

$$
u=\frac{(x_{13}-s)x_{23}}{x_{12}s}\frac{\xi z_{12}}{z_{23}(z_{13}-\xi)}=e\frac{x_{13}-s}{s}\,. \tag{3.13}
$$

Namely, taking $s=\frac{ex_{13}}{u+e}$ it can be shown that the exponents of $\xi,\bar{\xi}$ cancel and the integral can be rewritten as

$$
\begin{aligned}
A_3^{w=1,w=1}&=C_5(j_1,j_2,j_3)\gamma(j_2-j_1-j_3+1)\frac{\Gamma(j_1-J_1+\frac{k}{2})\Gamma(j_3-j_2+\bar{J}_1-\frac{k}{2})}{\Gamma(1-j_1+\bar{J}_1-\frac{k}{2})\Gamma(j_2-j_3-J_1+\frac{k}{2}+1)}\\
&\times 2i\pi(-1)^{m_1+\overline{m}_1}x_{12}^{J_3-J_1-j_2}\bar{x}_{12}^{\bar{J}_3-j_2-\bar{J}_1}x_{13}^{j_2-J_1-J_3}\bar{x}_{13}^{j_2-\bar{J}_1-\bar{J}_3}x_{23}^{J_1-j_2-J_3}\bar{x}_{23}^{\bar{J}_1-j_2-\bar{J}_3}\\
&\times z_{12}^{\Delta_3^w-\Delta_1^w-\Delta_2}\bar{z}_{12}^{\bar{\Delta}_3^w-\bar{\Delta}_1^w-\Delta_2}z_{23}^{-\Delta_2-\Delta_3^w+\Delta_1^w}\bar{z}_{23}^{-\Delta_2-\bar{\Delta}_3^w+\bar{\Delta}_1^w}z_{13}^{\Delta_2-\Delta_1^w-\Delta_3^w}\bar{z}_{13}^{\Delta_2-\bar{\Delta}_1^w-\bar{\Delta}_3^w}\\
&\times\lim_{\xi,\bar{\xi}\to 0}\int d^2u\ u^{j_2-j_3-J_1+\frac{k}{2}}\bar{u}^{j_2-j_3-\bar{J}_1+\frac{k}{2}}\\
&\quad\times(u+e)^{J_3+j_2-J_1}(\bar{u}+\bar{e})^{\bar{J}_3+j_2-\bar{J}_1}|1-u|^{2(j_1-j_2-j_3)}\\
&\quad\times\left[{}_2F_1\left(j_1+j_2-j_3,j_1-J_1+\frac{k}{2},j_2-j_3-J_1+\frac{k}{2}+1;u\right)\overline{{}_2F_1}(\bar{u})\right.\\
&\qquad+\lambda u^{j_3+J_1-j_2-\frac{k}{2}}\bar{u}^{j_3+\bar{J}_1-j_2-\frac{k}{2}}\\
&\qquad\left.\times{}_2F_1\left(j_1+j_3-j_2,j_1+J_1-\frac{k}{2},j_3+J_1-j_2-\frac{k}{2}+1;u\right)\overline{{}_2F_1}(\bar{u})\right],
\end{aligned}
$$

so that we can safely take the limit $\xi,\bar{\xi}\to 0$ inside the integral.

Thus we have found the functional form of the three point function but we still have to determine the j_i, J_i dependence. One way to do it is to solve the integral above. This can be done using results that have been found in reference [10], but it is very tedious. Therefore we will proceed along the alternative path, *i.e.* spectral flowing twice directly from the five point function.

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Since the coordinate dependence of the three point function has been determined we can use conformal invariance to fix three of the insertion points of the fields in A_5 whereas the other two get fixed from the spectral flow operation. In this way we find an integral which can be explicitly computed following [11, 12].

Let us fix $x_1 = z_1 = 0, x_2 = z_2 = 1, y_3 = \zeta_3 = \infty$ and set $y_1 = x_1 + t_1, \zeta_1 = z_1 + \epsilon_1, y_2 = x_2 + t_2$ and $\zeta_2 = z_2 + \epsilon_2$ in (3.6). Then the three point function takes the following form

$$
\begin{aligned}
&\left\langle \Phi^{w=1,j_1}_{J_1,\bar J_1}(0,0)\Phi^{w=1,j_2}_{J_2,\bar J_2}(1,1)\Phi_{j_3}(\infty,\infty)\right\rangle = \lim_{\epsilon_1,\epsilon_2,\bar\epsilon_1,\bar\epsilon_2\to0} \epsilon_1^{m_1}\bar\epsilon_1^{\overline{m}_1}\epsilon_2^{m_2}\bar\epsilon_2^{\overline{m}_2} \\
&\quad\times \int d^2t_1 d^2t_2\; t_1^{2(j_1-m_1-1)}\bar t_1^{2(j_1-\overline{m}_1-1)} t_2^{2(j_2-m_2-1)}\bar t_2^{2(j_2-\overline{m}_2-1)} A_5(t_1,\bar t_1,\epsilon_1,\bar\epsilon_1;t_2,\bar t_2,\epsilon_2,\bar\epsilon_2) \\
&= \lim_{\epsilon_1,\epsilon_2,\bar\epsilon_1,\bar\epsilon_2\to0} \epsilon_1^{m_1}\bar\epsilon_1^{\overline{m}_1}\epsilon_2^{m_2}\bar\epsilon_2^{\overline{m}_2} \int d^2t_1 d^2t_2\; t_1^{2(j_1-m_1-1)}\bar t_1^{2(j_1-\overline{m}_1-1)} t_2^{2(j_2-m_2-1)}\bar t_2^{2(j_2-\overline{m}_2-1)} \\
&\quad\times \left|\frac{\epsilon_1-t_1}{\epsilon_1(1+\epsilon_1)}\right|^{2(j_2-j_1-j_3)} \left|\frac{\epsilon_2-t_2}{\epsilon_2(1+\epsilon_2)}\right|^{2(j_1-j_2-j_3)} \left|\frac{t_1t_2}{\epsilon_1\epsilon_2}-\frac{(1+t_1)(1-t_2)}{(1+\epsilon_1)(1-\epsilon_2)}\right|^{2(j_3-j_1-j_2)} \\
&\quad\times|\epsilon_1(1+\epsilon_1)|^{-2j_1}|\epsilon_2(1+\epsilon_2)|^{-2j_2}|1+\epsilon_1-\epsilon_2|^{2(\Delta_3-\Delta_1-\Delta_2)}\,. \qquad (3.14)
\end{aligned}
$$

Performing the change of variables $t_1 = u\epsilon_1$, $t_2 = v\epsilon_2$, the exponents of ϵ_1 and ϵ_2 cancel and the integral becomes

$$
\begin{aligned}
&\int d^2u\, d^2v\; u^{j_1-m_1-1}\bar u^{j_1-\overline{m}_1-1}\bar v^{j_2-m_2-1}v^{j_2-\overline{m}_2-1} \\
&\qquad\qquad \times |u-1|^{2(j_2-j_1-j_3)}|v-1|^{2(j_1-j_2-j_3)}|uv-1|^{2(j_3-j_1-j_2)}\,.
\end{aligned}
$$

Defining $v' = v^{-1}$ this takes the form of the integral computed in reference [11] (see also [12]) which we review in appendix B for completeness. Reinserting the coordinate dependence, the final result is

$$
\begin{aligned}
&\left\langle \Phi^{w=1,j_1}_{J_1,\bar J_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar J_2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\right\rangle = B(j_1)B(j_2)C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_2,j_3\right) \\
&\quad\times\frac{\Gamma(j_3-J_1+J_2)\Gamma(j_3+\bar J_1-\bar J_2)\Gamma(2-j_1-j_2-j_3)^2}{\Gamma(1-j_3-J_1+J_2)\Gamma(1-j_3+\bar J_1-\bar J_2)}W(j_1,j_2,j_3,J_1,J_2,\bar J_1,\bar J_2) \\
&\quad\times x_{12}^{j_3-J_1-J_2}\bar x_{12}^{j_3-\bar J_1-\bar J_2}x_{13}^{(J_2-J_1-j_3)}\bar x_{13}^{\bar J_2-\bar J_1-j_3}x_{23}^{J_1-J_2-j_3}\bar x_{23}^{\bar J_1-\bar J_2-j_3} \\
&\quad\times z_{12}^{\Delta_3-\Delta_1^{w=1}-\Delta_2^{w=1}}\bar z_{12}^{\Delta_3-\bar\Delta_1^{w=1}-\bar\Delta_2^{w=1}}z_{13}^{\Delta_2^{w=1}-\Delta_1^{w=1}-\Delta_3}\bar z_{13}^{\bar\Delta_2^{w=1}-\bar\Delta_1^{w=1}-\Delta_3} \\
&\quad\times z_{23}^{\Delta_1^{w=1}-\Delta_2^{w=1}-\Delta_3}\bar z_{23}^{\bar\Delta_1^{w=1}-\bar\Delta_2^{w=1}-\Delta_3}\,, \qquad (3.15)
\end{aligned}
$$

where

$$
\begin{aligned}
W(j_i,J_i,\bar J_i) &= s(j_2-j_1-j_3)G\begin{bmatrix} j_2+J_2-\frac{k}{2}, j_2-j_1-j_3+1, 1-j_3+J_2-J_1 \\ j_2-j_1+J_2-J_1+1, 2-j_1-j_3+J_2-\frac{k}{2}\end{bmatrix} \\
&\times\left\{ s(j_1-j_2-j_3)G\begin{bmatrix} j_1-j_2-j_3+1, j_1+\bar J_1-\frac{k}{2}, 1-j_3+\bar J_1-\bar J_2 \\ 2-j_2-j_3+\bar J_1-\frac{k}{2}, j_1-j_2+\bar J_1-\bar J_2+1\end{bmatrix}\right. \\
&\left. -s(1-2j_2)G\begin{bmatrix} j_2-\bar J_2+\frac{k}{2}, j_2-j_1-j_3+1, 1-j_3+\bar J_1-\bar J_2 \\ j_2-j_1+\bar J_1-\bar J_2+1, 2-j_1-j_3-\bar J_2+\frac{k}{2}\end{bmatrix}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+s(j_1-j_2-j_3)G\left[\begin{matrix} j_1-j_2-j_3+1, j_1-J_1+\frac{k}{2}, 1-j_3-J_1+J_2 \\ 2-j_2-j_3-J_1+\frac{k}{2}, 1+j_1-j_2+J_2-J_1 \end{matrix}\right] \\
&\times\left\{-s(1-2j_1)G\left[\begin{matrix} j_1-j_2-j_3+1, j_1+\bar{J}_1-\frac{k}{2}, 1-j_3+\bar{J}_1-\bar{J}_2 \\ 2-j_2-j_3+\bar{J}_1-\frac{k}{2}, j_1-j_2+\bar{J}_1-\bar{J}_2+1 \end{matrix}\right]\right. \\
&\left.+s(j_2-j_1-j_3)G\left[\begin{matrix} j_2-\bar{J}_2+\frac{k}{2}, j_2-j_1-j_3+1, 1-j_3+\bar{J}_1-\bar{J}_2 \\ j_2-j_1+\bar{J}_1-\bar{J}_2+1, 2-j_1-j_3-\bar{J}_2+\frac{k}{2} \end{matrix}\right]\right\},
\end{aligned} \tag{3.16}
$$

with $G\left[\begin{smallmatrix} a,b,c \\ e,f \end{smallmatrix}\right] \equiv \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}\,{}_3F_2(a,b,c;e,f;1)$ and $s(a) = \sin(\pi a)$. The expression for $W(j_i, J_i, \bar{J}_i)$ can be rewritten using standard properties of the generalized hypergeometric function listed in appendix B to show that, since $m_i - \overline{m}_i \in \mathbb{Z}$, it is symmetric with respect to J_i and $\bar{J}_i$.

As a consistency check on the result (3.15), we verify that it reduces to the two point function of one unit spectral flowed states when $j_3 = 0$. Indeed we obtain

$$
\begin{aligned}
\left\langle \Phi^{w=1,j_1}_{J,\bar{J}}(x_1,z_1)\Phi^{w=1,j_2}_{J,\bar{J}}(x_2,z_2)\Phi_{j_3=0}(x_3,z_3)\right\rangle &= x_{12}^{-2J}\bar{x}_{12}^{-2\bar{J}}z_{12}^{-2\Delta_1^{w=1}}\bar{z}_{12}^{-2\bar{\Delta}_1^{w=1}} \\
\times B(j_1)\delta(j_1-j_2)\Gamma(0)&\frac{\Gamma(1-2j_1)}{\Gamma(2j_1)}\frac{\Gamma(j_1+m_1)\Gamma(j_1-\bar{m}_1)}{\Gamma(1-j_1+m_1)\Gamma(1-j_1-\bar{m}_1)},
\end{aligned}
$$

in agreement with the results in [1]. Here we have used the properties of $B(j)$ and $C(j_1, j_2, j_3)$ which are listed in appendix A. We can identify the factor $\Gamma(0)$ in this expression with the volume of the conformal group of S^2 with two fixed points, namely $V_{\rm conf} = \int d^2z|z|^{-2}$. Actually, as discussed in reference [1], in general a divergence arises when computing correlators which include spectral flowed fields. The definition (2.5) of these fields already contains a rescaling $\tilde{\Phi} \to \Phi = V_{\rm conf}\tilde{\Phi}$, thus in this case there is a factor $V^2_{\rm conf}$ whose product with $V^{-1}_{\rm conf}$ in the expression (5.13) of reference [1], computed in the m basis, explains the factor $V_{\rm conf} \sim \Gamma(0)$ which we have found here.

Let us analyze the properties of our result. The function $W(j_i, J_i, \bar{J}_i)$ is analytic in its arguments for states belonging to the continuous representation or their spectral flow images. Therefore the three point function (3.15) is perfectly well behaved and finite for normalizable operators with $j = \frac{1}{2} + is$, as expected. If one of the original unflowed states, say Φ_{j_1}, belongs to a lowest weight representation, *i.e.*, $m_1 = j_1 + n_1, \overline{m}_1 = j_1 + \overline{n}_1$ with $n_1, \bar{n}_1 = 0, 1, 2, \ldots$, then it can be shown that $W(j_i, J_i, \bar{J}_i)$ greatly simplifies, and taking further $n_1, \bar{n}_1 = 0$ the hypergeometric functions become unity. The analysis of $W(j_i, J_i, \bar{J}_i)$ completely agrees with that of reference [12] (taking into account the change in notation). However notice that we are dealing with a winding conserving three point function which includes two one unit spectral flowed states whereas [12] considers unflowed states. Moreover (3.15) is an x basis correlator unlike the m basis expression analyzed in [12].

The three point function (3.15) has various poles which come from the poles in C_5, in the $\Gamma-$functions and in the unrenormalized hypergeometric functions. C_5 has the same poles as the unflowed three point function, namely at

$$
j = n + m(k-2), \qquad -(n+1)-(m+1)(k-2), \qquad n, m = 0, 1, 2, \ldots\,, \tag{3.17}
$$

with

$$j = 1 - j_1 - j_2 - j_3, \quad j_1 - j_2 - j_3, \quad j_2 - j_3 - j_1, \quad j_3 - j_2 - j_1 \ . \tag{3.18}$$

The Γ−functions add the following poles

$$J_1 = J_2 + j_3 + n, \quad J_2 = J_1 + j_3 + n \,, \tag{3.19}$$

and similar ones for $\bar{J}_1, \bar{J}_2$. The poles of $G\left[\begin{smallmatrix} a,b,c \\ e,f \end{smallmatrix}\right]$ are at $a, b, c, u = -n$, with $u = e + f - a - b - c$, and thus they are all contained in the previous ones except for the poles signaling the presence of spectral flowed images of the discrete representations, *e.g.* $m_1 = j_1 + n_1, \overline{m}_1 = j_1 + \overline{n}_1$. Therefore the pole structure is as discussed in reference [1] in the unflowed case with the addition of (3.19), which are analogous to poles in the S matrix of string theory in Minkowski space.

4. Ward identities, KZ and null vector equations

The computation of more complicated correlation functions along the lines of the previous section would require to start from higher point amplitudes. Actually the cases following in complexity, namely the three point function including three one-unit spectral flowed operators or the four point function involving one $w = 1$ field require the knowledge of the six point function with three spectral flow operators and three physical states or the five point function with one $\Phi_{\frac{k}{2}}$ and four generic unflowed fields respectively. In this section we discuss general properties of correlation functions containing $w = 1$ spectral flowed operators in the x basis in order to find an alternative method to compute such more complicated amplitudes. More specifically, we will derive the Ward identities and the modified KZ and null vector equations to be satisfied by generic correlators including $w = 1$ fields. We begin by giving an account of the already known results on the subject and the difficulties we expect to find in order to make further progress.

4.1 Ward identities

We investigate first the form of the Ward identities when the definitions (2.5) or (2.7) are used for the $w = 1$ field. This can also be understood as an additional consistency check on such definition.

Let us start by considering N point functions of primary $w = 0$ fields,

$$A_N \equiv \langle \Phi_{j_1}(x_1, z_1) \Phi_{j_2}(x_2, z_2) \cdots \Phi_{j_N}(x_N, z_N) \rangle \ .$$

It is well known that the global SL(2) symmetry of the WZW model determines the Ward identities to be satisfied by the correlation functions, namely

$$0 = \sum_{i=1}^{N} \frac{\partial A_N}{\partial x_i} \,, \tag{4.1}$$

$$0 = \sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) \, A_N \,, \tag{4.2}$$

$$0 = \sum_{i=1}^{N} \left(x_i^2 \frac{\partial}{\partial x_i} + 2 j_i x_i \right) \, A_N \ . \tag{4.3}$$

Now suppose we consider an $N+1$ point function including one spectral flow operator $\Phi_{\frac{k}{2}}$ at position (x_2, z_2),

$$A_{N+1} \equiv \langle \Phi_{j_1}(x_1, z_1) \Phi_{\frac{k}{2}}(x_2, z_2) \cdots \Phi_{j_{N+1}}(x_{N+1}, z_{N+1}) \rangle \ , \tag{4.4}$$

and take

$$x_1 = x_2 + y \ , \qquad z_1 = z_2 + \epsilon \ . \tag{4.5}$$

In order to obtain the differential equations determined by the Ward identities one has to be careful that the derivatives act only once on each of the points where the fields are inserted. Therefore it is convenient to perform the following transformation

$$\frac{\partial}{\partial x_1} \longrightarrow \frac{\partial}{\partial y} \ , \qquad \frac{\partial}{\partial x_2} \longrightarrow \frac{\partial}{\partial x_2} - \frac{\partial}{\partial y} \ , \tag{4.6}$$

so that the derivatives with respect to x_2 act only on the field in the second position and not on the first one. The Ward identities transform accordingly, so for instance equation (4.2) reads

$$\begin{aligned} 0 &= \sum_{i=1}^{N+1} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) A_{N+1} \\ &= \left[(x_2 + y) \frac{\partial}{\partial y} + x_2 \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y} \right) + j_1 + \frac{k}{2} + \sum_{i=3}^{N+1} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) \right] A_{N+1} \\ &= \left[y \frac{\partial}{\partial y} + x_2 \frac{\partial}{\partial x_2} + j_1 + \frac{k}{2} + \sum_{i=3}^{N+1} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) \right] A_{N+1} \ . \end{aligned} \tag{4.7}$$

We want to derive the Ward identities for amplitudes containing $w = 1$ spectral flowed operators. Equation (2.5) suggests to apply the following operation on (4.7)

$$\lim_{\epsilon, \bar{\epsilon} \to 0} \epsilon^m \bar{\epsilon}^{\bar{m}} \int d^2 y \ y^{j_1 - m - 1} \bar{y}^{j_1 - \bar{m} - 1} \ . \tag{4.8}$$

Integrating by parts, it can be seen that the Ward identity (4.7) turns into

$$\begin{aligned} 0 &= \left[-(j_1 - m) + x_2 \frac{\partial}{\partial x_2} + j_1 + \frac{k}{2} + \sum_{i=3}^{N+1} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) \right] A_N^w \\ &= \left[x_2 \frac{\partial}{\partial x_2} + \left(m + \frac{k}{2} \right) + \sum_{i=3}^{N+1} \left(x_i \frac{\partial}{\partial x_i} + j_i \right) \right] A_N^w \ , \end{aligned} \tag{4.9}$$

where from the definition (2.5) we identify

$$A_N^w \equiv \left\langle \Phi^{w=1, j_1}_{m+\frac{k}{2}, \bar{m}+\frac{k}{2}}(x_2, z_2) \Phi_{j_3}(x_3, z_3) \cdots \Phi_{j_{N+1}}(x_{N+1}, z_{N+1}) \right\rangle \ . \tag{4.10}$$

Notice that equation (4.9) is precisely of the same form as (4.2) with the identification (2.6) for the spin of the spectral flowed field. It can be shown that the same procedure gives an equivalent result for the two other Ward identities (4.1) and (4.3).

Now we focus on global conformal invariance, which determines the following differential equations for the correlators

$$0 = \sum_{i=1}^{N} \frac{\partial A_N}{\partial z_i} \ , \tag{4.11}$$

$$0 = \sum_{i=1}^{N} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \ A_N \ , \tag{4.12}$$

$$0 = \sum_{i=1}^{N} \left(z_i^2 \frac{\partial}{\partial z_i} + 2\Delta_i z_i \right) \ A_N \ , \tag{4.13}$$

where the factors Δ_i are the conformal dimensions of the fields (see (2.2)). In order to derive the corresponding Ward identities for correlation functions containing one $w = 1$ state we repeat the steps discussed above with the change of variables (4.5) and the corresponding transformation for the derivatives

$$\frac{\partial}{\partial z_1} \longrightarrow \frac{\partial}{\partial \epsilon} \ , \qquad \frac{\partial}{\partial z_2} \longrightarrow \frac{\partial}{\partial z_2} - \frac{\partial}{\partial \epsilon} \ . \tag{4.14}$$

Then for instance eq. (4.12) becomes

$$\begin{aligned} 0 &= \sum_{i=1}^{N+1} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \ A_{N+1} \\ &= \left[(z_2 + \epsilon)\frac{\partial}{\partial \epsilon} + z_2 \left(\frac{\partial}{\partial z_2} - \frac{\partial}{\partial \epsilon} \right) + \Delta_1 - \frac{k}{4} + \sum_{i=3}^{N+1} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \right] \ A_{N+1} \\ &= \left[\epsilon \frac{\partial}{\partial \epsilon} + z_2 \frac{\partial}{\partial z_2} + \Delta_1 - \frac{k}{4} + \sum_{i=3}^{N+1} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \right] \ A_{N+1} \ . \end{aligned} \tag{4.15}$$

Applying to this equation the operation

$$\lim_{y,\bar{y} \to 0} y^{j_1 - m} \bar{y}^{j_1 - \bar{m}} \int d^2\epsilon \ \epsilon^{m-1} \bar{\epsilon}^{\bar{m}-1} \ , \tag{4.16}$$

which is suggested by eq. (2.7), and performing an integration by parts, we can see that A_N^w satisfies the spectral flowed Ward identity

$$0 = \left[z_2 \frac{\partial}{\partial z_2} + \left(\Delta_1 - m - \frac{k}{4} \right) + \sum_{i=3}^{N+1} \left(z_i \frac{\partial}{\partial z_i} + \Delta_i \right) \right] \ A_N^w \ ,$$

which is of the same form as (4.12) with the following identification for the conformal dimension of the $w = 1$ field

$$\Delta_1^{w=1} = \Delta_1 - m - \frac{k}{4} = \Delta_1 - J + \frac{k}{4} \ , \tag{4.17}$$

in agreement with (2.3). A similar expression can be found for $\bar{\Delta}_1^{w=1}$ in terms of $\bar{J}$. Again, all this goes through for the two other equations (4.11) and (4.13).

Therefore we conclude that the Ward identities to be satisfied by correlation functions including the operator $\Phi^{w=1,j}_{J,\bar{J}}$ coincide with those of the unflowed case with the modifications (2.6) and (4.17) for the spin and conformal weight of the $w = 1$ field respectively. This analysis can be generalized to correlation functions including an arbitrary number of $w = 1$ states. From here the general form of the two and three point functions containing $w = 1$ fields is completely determined, whereas the four point functions depend, as usual, on the anharmonic ratios.

4.2 Modified KZ and null vector equations

Now we want to determine the form that the KZ and null vector equations take for correlators including $w = 1$ fields. In order to do this, let us consider again the $N + 1$-point function (4.4). Consider *e.g.* any point z_i with $i \geq 3$. The correlator A_{N+1} satisfies the standard KZ equation of the form

$$(k-2)\,\frac{\partial A_{N+1}}{\partial z_i} = \sum_{n=1,\, n\neq i}^{N+1} \frac{1}{z_i - z_n}\left[(x_n - x_i)^2 \frac{\partial^2}{\partial x_i \partial x_n} + \right.$$
$$\left. +2(x_n - x_i)\left(j_n \frac{\partial}{\partial x_i} - j_i \frac{\partial}{\partial x_n}\right) - 2 j_i j_n\right] A_{N+1}\,. \tag{4.18}$$

In addition, since the spectral flow operator at (x_2, z_2) has a null descendant, namely $J^-_{-1}|j = k/2; m = k/2\rangle = 0$, then A_{N+1} must also obey the following null vector equation

$$0 = \sum_{n=1,\, n\neq 2}^{N+1} \frac{x_n - x_2}{z_2 - z_n}\left[(x_n - x_2)\frac{\partial}{\partial x_n} + 2 j_n\right] A_{N+1}\,. \tag{4.19}$$

Our aim here is to perform similar manipulations to those in the previous subsection, in order to investigate the form of the equations to be satisfied by the N point function including one $w = 1$ field, namely A^w_N in (4.10). The general idea is that (4.10) can be obtained from (4.4) by performing the fusion of $\Phi_{j_1}(x_1, z_1)$ with $\Phi_{k/2}(x_2, z_2)$ through the prescription (2.5). In that way, the equations to be satisfied by A_{N+1}, namely (4.18) and (4.19), are expected to turn into those to be obeyed by A^w_N.

In order to do this, let us start by performing the change of variables (4.5) in the KZ equation (4.18) which can then be rewritten as

$$(k-2)\,\frac{\partial A_{N+1}}{\partial z_i} = \frac{1}{z_i - z_2 - \epsilon}\left[(x_2 + y - x_i)^2 \frac{\partial^2}{\partial x_i \partial y}\right.$$
$$\left. + 2\,(x_2 + y - x_i)\left(j_1 \frac{\partial}{\partial x_i} - j_i \frac{\partial}{\partial y}\right) - 2 j_i j_1\right] A_{N+1}$$
$$+\frac{1}{z_i - z_2}\left[(x_2 - x_i)^2\left(\frac{\partial^2}{\partial x_i \partial x_2} - \frac{\partial^2}{\partial x_i \partial y}\right)\right.$$
$$\left. + 2\,(x_2 - x_i)\left(\frac{k}{2}\frac{\partial}{\partial x_i} - j_i \frac{\partial}{\partial x_2} + j_i \frac{\partial}{\partial y}\right) - k j_i\right] A_{N+1}$$

$$+\sum_{n=3,\, n\neq i}^{N+1}\frac{1}{z_i-z_n}\left[(x_n-x_i)^2\frac{\partial^2}{\partial x_i\partial x_n}+\right.$$
$$\left.+\,2(x_n-x_i)\left(j_n\frac{\partial}{\partial x_i}-j_i\frac{\partial}{\partial x_n}\right)-2j_ij_n\right]A_{N+1}\,. \quad (4.20)$$

Now acting with the operator (4.8) and integrating by parts in y, we obtain the modified KZ equation for A_N^w, namely

$$\begin{aligned}(k-2)\,\frac{\partial A_N^w(J)}{\partial z_i} &= -\left(j_1-J+\frac{k}{2}-1\right)\,\frac{x_2-x_i}{(z_i-z_2)^2}\,\left[(x_2-x_i)\frac{\partial}{\partial x_i}-2j_i\right]\,A_N^w(J+1)\\ &\quad+\frac{1}{z_i-z_2}\left[(x_2-x_i)^2\frac{\partial^2}{\partial x_i\partial x_2}+\right.\\ &\qquad\left.+\,2(x_2-x_i)\left(J\frac{\partial}{\partial x_i}-j_i\frac{\partial}{\partial x_2}\right)-2j_iJ\right]\,A_N^w(J)\\ &\quad+\sum_{n=3,\, n\neq i}^{N+1}\frac{1}{z_i-z_n}\left[(x_n-x_i)^2\frac{\partial^2}{\partial x_i\partial x_n}+\right.\\ &\qquad\left.+\,2(x_n-x_i)\left(j_n\frac{\partial}{\partial x_i}-j_i\frac{\partial}{\partial x_n}\right)-2j_ij_n\right]\,A_N^w(J)\,, \end{aligned} \quad (4.21)$$

where J is the spin of the $w=1$ field, given by (2.6).

The notation $A_N^w(J+1)$ indicates that we must replace $J \longrightarrow J+1$ in A_N^w. Thus, eq. (4.21), which is interpreted as the KZ equation for an N point function including one $w=1$ field, differs from the standard KZ equation for correlators of unflowed fields. In fact, notice that (4.21) is an *iterative* relation in the spin of the spectral flowed field. As we will see, the property of being iterative in the spins of the spectral flowed fields will be common to all the equations to be satisfied by correlators including fields in $w\neq 0$ sectors. In fact, such a novel feature is not surprising, since it is inherited from the primary state condition (2.8). We also point out that an equation analogous to (4.21) holds for the antiholomorphic part, where the iterative variable is $\bar{J}$ (see (2.6)).

Now, following a similar procedure with the null vector equation (4.19) we obtain an additional iterative equation, namely

$$\left(j_1+J-\frac{k}{2}-1\right)\,A_N^w(J-1)=\sum_{n=3}^{N+1}\frac{x_n-x_2}{z_2-z_n}\left[(x_n-x_2)\frac{\partial}{\partial x_n}+2j_n\right]\,A_N^w(J)\,, \quad (4.22)$$

which is understood as the modified null vector equation to be satisfied by correlators containing one $w=1$ field. It supplements (4.21), so that both equations must be solved in order to find the explicit expression of A_N^w. As before, an analogous equation holds for the antiholomorphic part, with $\bar{J}$ as the iterative variable. The procedure detailed here can be extended to the case of correlators including any number of $w=1$ fields, where the spins of all the spectral flowed fields turn out to be iterative variables. In the following section we will consider some specific calculations.

5. Four point function including one $w = 1$ field

The purpose of this section is to explicitly solve the modified KZ and null vector equations corresponding to the four point function involving one $w = 1$ field, namely

$$A_4^w = \left\langle \Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2)\Phi_{J,\bar{J}}^{w=1,j_3}(x_3, z_3)\Phi_{j_4}(x_4, z_4)\right\rangle \ , \tag{5.1}$$

which, according to the prior discussions, can be obtained from the five point function

$$\left\langle \Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2)\Phi_{j_3}(y, \zeta)\Phi_{\frac{k}{2}}(x_3, z_3)\Phi_{j_4}(x_4, z_4)\right\rangle \ , \tag{5.2}$$

through the prescription (2.5).

From the results of the previous section, we expect that A_4^w had the same functional form as an unflowed four point function, but with the spin and conformal dimension of the $w = 1$ field given by $J = m + \frac{k}{2}$, $\Delta_3^{w=1} = \Delta_3 - J + \frac{k}{4}$. Thus, we consider the following expression for A_4^w

$$\begin{aligned} A_4^w = & \int dj\ B(j_3)C(j_1, j_2, j)\ B(j)^{-1}C\left(j, \frac{k}{2} - j_3, j_4\right) \\ & \times D_1(j_1, j_2, j_3, J, j_4, j)\ D_2(j_1, j_2, j_3, \bar{J}, j_4, j)\ \mathcal{F}(z, x)\ \bar{\mathcal{F}}(\bar{z}, \bar{x}) \\ & \times \left(x_{43}^{j_1+j_2-j_4-J} x_{42}^{-2j_2} x_{41}^{J+j_2-j_4-j_1} x_{31}^{j_4-j_1-j_2-J}\right) \\ & \times \left(z_{43}^{\Delta_1+\Delta_2-\Delta_4-\Delta_3^{w=1}} z_{42}^{-2\Delta_2} z_{41}^{\Delta_3^{w=1}+\Delta_2-\Delta_4-\Delta_1} z_{31}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3^{w=1}}\right) \\ & \times (\text{antiholomorphic part})\ , \end{aligned} \tag{5.3}$$

where the dependence in the coefficients B and C is inherited from the five point function (5.2) (see details in appendix A). Notice that, due to the presence of the spectral flow operator, we are left with only one state in one of the two intermediate channels. The other one contributes the integral in j which has to be performed over $\frac{1}{2} + i\mathbb{R}$.

In addition, D_1 and D_2 are the parts of the coefficient of the four point function depending respectively on the right and left spins of the string states, whereas $\mathcal{F}$ and $\bar{\mathcal{F}}$ are functions of the cross ratios

$$z = \frac{z_{21}z_{43}}{z_{31}z_{42}}\ , \qquad x = \frac{x_{21}x_{43}}{x_{31}x_{42}}\ . \tag{5.4}$$

Now plugging (5.3) into the modified KZ and null vector equations (which, up to the obvious change in labels, are given by (4.21) and (4.22)) we find respectively the following iterative expressions[3]

$$\begin{aligned} &\left(j_3 - J + \frac{k}{2} - 1\right)\left[(1-x)\frac{\partial}{\partial x} + (j_1 + j_4 - j_2 - J - 1)\right] zD_1(J+1)\mathcal{F}_{J+1} = \\ &= \Bigg\{ -(k-2)z(1-z)\frac{\partial}{\partial z} + x(1-x)(z-x)\frac{\partial^2}{\partial x^2} - \Big[(j_4 - j_1 - j_2 + J - 1)z \\ &\qquad -2(j_4 - j_2 - 1)xz + 2(j_1 + j_2)x + (j_4 - j_1 - 3j_2 - J - 1)x^2\Big]\frac{\partial}{\partial x} \end{aligned}$$

[3]This is the modified KZ equation at (x_1, z_1).

$$+\left[2j_2(j_4-j_2)-j_3(j_3-1)+J(J-1)-(k-2)J+k(k-2)/4\right]\, z$$
$$\left.-2j_2(j_4-j_1-j_2-J)x-2j_1j_2\right\}D_1(J)\,\mathcal{F}_J\ , \tag{5.5}$$

and

$$\left(j_3+J-\frac{k}{2}-1\right)(1-z)D_1(J-1)\mathcal{F}_{J-1} \tag{5.6}$$
$$=\left[(j_2-j_4-j_1+J)(1-z)-2j_2(1-x)+(1-x)(z-x)\frac{\partial}{\partial x}\right]D_1(J)\mathcal{F}_J\ .$$

Here $D_1(J\pm1)$ ($\mathcal{F}_{J\pm1}$) indicates that we must replace $J\longrightarrow J\pm1$ in D_1 ($\mathcal{F}$). Analogous expressions can be found for the antiholomorphic part $D_2(\bar{J})$ ($\bar{\mathcal{F}}_{\bar{J}}$).

Now we follow a similar route to that in the unflowed case [1, 3] and expand $\mathcal{F}$ in powers of z as follows

$$\mathcal{F}(z,x)=z^{\Delta_j-\Delta_{j_1}-\Delta_{j_2}}\ x^{j-j_1-j_2}\sum_{n=0}^{\infty}f_n(x)z^n\ . \tag{5.7}$$

We then focus on the lowest order of this expansion. We consider first the KZ equation (5.5). The crucial result is that, to the lowest order in z, the iterative term in the l.h.s. does not contribute, whereas the r.h.s. reduces to an expression of precisely the same form as that of the lowest order of the standard (unflowed) KZ equation, as computed in [3], with the only difference that j_3 is replaced by J. Thus we have the following solution[4]

$$f_0={}_2F_1(j-j_1+j_2,j+J-j_4,2j;x)\ , \tag{5.8}$$

where ${}_2F_1$ is the standard hypergeometric function.

Now we turn to the modified null vector equation (5.6). Keeping the lowest order in z and using (5.8) we find that the coefficient D_1 must satisfy the following iterative relation

$$\left(j_3+J-\frac{k}{2}-1\right)\ D_1(J-1)\ =\ (J-j-j_4)\ D_1(J)\ , \tag{5.9}$$

with an analogous expression for D_2. This allows to cancel the coefficient D_1 in (5.5) and (5.6) and we can write equations for all higher order terms in (5.7) starting from the lowest order (5.8), *i.e.* we are able to find iterative equations for f_n in terms of f_{n-1} (for $n\geq1$). This is done by plugging (5.7) into (5.5) and (5.6) and using (5.9). For instance the modified KZ equation (5.5) gives

$$(j_4-J+j-1)\left\{x^2(1-x)\frac{d^2}{dx^2}+\left[(j_1-j_2+j_4-J-2j-1)x+2j\right]x\frac{d}{dx}\right.$$

[4]Actually the solution is a linear combination of the functions ${}_2F_1(j-j_1+j_2,j+J-j_4,2j;x)$ and $x^{1-2j}{}_2F_1(1-j-j_1+j_2,1-j+J-j_4,2-2j;x)$. However, analogously as in the unflowed case [1], we may use the fact that, when inserted in (5.7), the two solutions are related to each other through the symmetry $j\longrightarrow1-j$ which allows to keep only the first solution provided that in (5.3) we now integrate j over the entire imaginary axis, i.e. $\frac{1}{2}+i\mathbf{R}$.

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$$
\begin{aligned}
&+[(j_1+j_2-j)(2j_2-j_4+J+j)-2j_2(j_1+j_2-j_4+J)]x+(k-2)n\Big\}\; f_n^{(J)} = \\
&\quad = \left(j_3+J-\frac{k}{2}\right)\left(j_3-J+\frac{k}{2}-1\right) \\
&\quad \times\left[(1-x)\frac{d}{dx}-(j_1+j_2-j)\frac{1-x}{x}+j_1-j_2+j_4-J-1\right]\; f_{n-1}^{(J+1)} \\
&\quad +(j_4-J+j-1)\Big\{x(1-x)\frac{d^2}{dx^2} \\
&\quad +[2(j_1+j_4-j-1)x-(j_1+j_2+j_4+J-2j-1)]\,\frac{d}{dx} \\
&\quad +(j_1+j_2-j)(j_4+J-j)\frac{1}{x}-2j_1j_4-j_3(j_3-1) \\
&\quad +2j(j_1+j_4-j)+J(J-1)+(k-2)(n-1-J+k/4)\Big\}\; f_{n-1}^{(J)}\,. \qquad (5.10)
\end{aligned}
$$

An interesting thing about this equation is that, even when it is also iterative in J, as expected, such iterative terms are all related to the $(n-1)$-order factor, f_{n-1}, whereas there is no iterative term for f_n. This allows to write an equation for $f_n^{(J)}$ in terms of the data $f_{n-1}^{(J)}$ and $f_{n-1}^{(J+1)}$.

A similar procedure can be followed with the modified null vector equation (5.6) and we obtain

$$
\begin{aligned}
&\left[x(1-x)\frac{d}{dx}+(j_1-j_2-j)x+j_4+j-J\right]\; f_n^{(J)}+(J-j-j_4)\; f_n^{(J-1)} \qquad (5.11) \\
&=\left[(1-x)\frac{d}{dx}+j_4+2j_1-j-J+(j-j_1-j_2)\,\frac{1}{x}\right]\; f_{n-1}^{(J)}+(J-j-j_4)\; f_{n-1}^{(J-1)}\,.
\end{aligned}
$$

The calculations we have performed so far are similar in spirit to those considered in the unflowed case [3] (see also [1]). Notice, however, that we have not completely determined the coefficient D_1 yet. Even though the functional dependence on the coordinates is fixed, all the information we have to fully determine the spin dependent coefficient is the iterative expression (5.9). This means that the modified KZ and null vector equations do not completely specify the spin dependence of the four point function. This is not surprising since a similar situation is found in the unflowed case. Nevertheless, we are still able to find a proper expression for the coefficient by requiring the following two conditions: *i*) that it satisfies (5.9) (and a similar expression for D_2), and *ii*) that A_4^w in (5.3) correctly reduces to (2.12), the three point function involving one spectral flowed field, when one of the unflowed operators is the identity.

It can be shown that a solution to *i*) and *ii*) is given by

$$
D_1D_2 \sim \frac{1}{\gamma\left(j_1+j_2+j_3+j_4-\frac{k}{2}\right)}\,\frac{\Gamma\left(j_3+J-\frac{k}{2}\right)}{\Gamma(1+J-j_4-j)}\,\frac{\Gamma(j_4+j-\bar{J})}{\Gamma\left(1-j_3-\bar{J}+\frac{k}{2}\right)}\,, \qquad (5.12)
$$

up to a k dependent coefficient. Requirement *i*) can be verified using standard properties of Γ-functions, whereas *ii*) is also satisfied since, using the expression above together with (5.8) and some of the identities in appendix A, it can be shown that (5.3) reduces to (2.12) for $j_2=0$.

We should point out however that the solution (5.12) is not unique. For instance, the coefficient in (3.10) verifies both requirements *i*) and *ii*) but it does not match (5.12).[5] Such residual uncertainties might be removed studying the factorization properties of the four point function (5.3), following a similar path to that of section 4 in reference [1] for the unflowed case. However here the pole structure of the four point function presents additional difficulties since there are poles in the integral in the complex j plane crossing the integration contour even before performing the analytic continuation. Therefore we leave this analysis for future work.

6. Discussion and conclusions

The purpose of this work was to study correlation functions involving one unit spectral flowed string states in AdS_3. We have computed the three point function including one unflowed and two $w = 1$ states in the WZW model in SL(2,$\mathbb{R}$), and performed the analysis of the corresponding pole structure. We have also considered the four point function with one $w = 1$ and three generic $w = 0$ states.

We performed various checks on our results. In particular, the Ward identities prescribing the general form of correlation functions containing spectral flowed fields were discussed and we then verified that the three and four point functions computed indeed have the form dictated by conformal and global SL(2) invariance. In addition, we also verified that the three point function including two $w = 1$ operators reproduces the corresponding two point function of $w = 1$ spectral flowed fields when the third operator is the identity.

Our results represent one step forward towards establishing the consistency of string theory on AdS_3. This would require the analysis of the factorization properties of the four point function (5.3). Actually the structure of the factorization of the unflowed four point function contains several differences with the flat case and it would be interesting to see how they generalize when winding is considered. Indeed, it was argued in reference [1] that the four point functions do not factorize as expected into sums of products of three point functions with physical intermediate states unless the quantum numbers of the external states verify $j_1 + j_2 < \frac{k+1}{2}, j_3 + j_4 < \frac{k+1}{2}$. The interpretation of these constraints presented in [1] indicates that correlation functions violating these bounds do not represent well defined computations in the dual CFT description of the theory on the boundary. This explanation is similar to the interpretation of the upper bound on the spin of the physical states (*i.e.*, $j < \frac{k-1}{2}$) as the condition that only local operators be considered in the boundary CFT. However in the later case one has a clear understanding of the constraint from the representations of SL(2,$\mathbb{R}$) which define the theory in the bulk. Similarly one would like to better understand this unusual feature of the correlation functions from the worldsheet viewpoint. Moreover the factorization structure of the four point function (5.3)

[5] When checking these last statements it must be taken into account that (3.10) involves, apart from the obvious changes in labels and the presence of a spectral flow operator, a result which is expressed in cross ratios other than those in (5.4), so that appropriate transformations must be performed on (3.10) before comparing it to the expressions in this section.

would also be important to unambiguously determine the spin dependent coefficient $D_1 D_2$. However, as discussed in section 5, the factorization of A_4^w presents additional difficulties to those encountered in the unflowed case since there are poles in the integral in the complex j plane crossing the integration contour even before performing the analytic continuation. Therefore we postpone the analysis of the consistency of string theory when spectral flowed correlators are considered for future work.

Correlation functions involving states in higher winding sectors have not been considered so far. This is an important ingredient for the complete determination of string theory on AdS$_3$. However this would require, along the lines we have presented here, first of all a proper definition of such fields in the x basis, which is not known yet.

We have obtained the modified KZ and null vector equations to be satisfied by correlation functions containing $w = 1$ spectral flowed fields.[6] We have shown that these are iterative equations relating amplitudes generically involving spectral flowed fields with spins J, $J+1$ and $J-1$. We managed to manipulate these expressions and analyze the four point function containing one $w = 1$ field. The modified KZ and null vector equations also allowed us to obtain certain three point functions involving three $w = 1$ fields for particular combinations of spins (see appendix C). A similar analysis for the three point function containing two $w = 1$ states shows, when comparing with the procedure followed in section 3, that the specific spin relations found correspond to simplified integrals in the spectral flowing procedure of the original higher point unflowed function involving the operators $\Phi_{\frac{k}{2}}$, and therefore they seem to have no physical relevance. A more general *ansatz* than the one we proposed in (C.12), possibly involving hypergeometric functions, would be required in order to obtain the full three point function.

Finally let us observe that, according to the well known relation between correlation functions in the SL(2,$\mathbb{R}$) WZW model and Liouville theory [7, 14], our results can also be used to obtain amplitudes in this later theory.[7]

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A. Correlators containing $\Phi_{\frac{k}{2}}$

In this appendix we obtain the j_i dependent coefficients for the five and six point functions respectively containing two and three spectral flow operators $\Phi_{\frac{k}{2}}$ as well as three operators

[6] See [13] for a different approach to the construction of the KZ equations for correlation functions containing spectral flowed fields.

[7] See [13, 15] for more recent work on this connection.

of generic spins j_1, j_2, j_3, in the limit where each $\Phi_{\frac{k}{2}}$ fuses with a Φ_j to give a spectral flowed operator. That is, starting from

$$\langle \Phi_{j_1} \Phi_{\frac{k}{2}} \Phi_{j_2} \Phi_{\frac{k}{2}} \Phi_{j_3} \rangle \ , \tag{A.1}$$

and

$$\langle \Phi_{j_1} \Phi_{\frac{k}{2}} \Phi_{j_2} \Phi_{\frac{k}{2}} \Phi_{j_3} \Phi_{\frac{k}{2}} \rangle \ , \tag{A.2}$$

we would like to respectively obtain the j_i dependent coefficient of the following three point functions

$$\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1} \Phi^{w=1,j_2}_{J_2,\bar{J}_2} \Phi_{j_3} \rangle \ , \tag{A.3}$$

and

$$\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1} \Phi^{w=1,j_2}_{J_2,\bar{J}_2} \Phi^{w=1,j_3}_{J_3,\bar{J}_3} \rangle \ . \tag{A.4}$$

In addition, we will perform analogous calculations for the five point function including only one $\Phi_{\frac{k}{2}}$, namely

$$\langle \Phi_{j_1} \Phi_{j_2} \Phi_{j_3} \Phi_{\frac{k}{2}} \Phi_{j_4} \rangle \ , \tag{A.5}$$

in order to obtain the j_i dependent coefficient of the four point function

$$\langle \Phi_{j_1} \Phi_{j_2} \Phi^{w=1,j_3}_{J,\bar{J}} \Phi_{j_4} \rangle \ . \tag{A.6}$$

The following properties of the B and C coefficients of the two and three point functions will be useful, namely [1]

$$\begin{aligned}
B\left(\frac{k}{2}-j\right) &\sim \frac{1}{B(j)} \ , && \text{(A.7)}\\
C\left(j_1, j_2, \frac{k}{2}\right) &\sim \delta\left(j_1 + j_2 - \frac{k}{2}\right) \ , && \text{(A.8)}\\
C\left(\frac{k}{2}-j, \frac{k}{2}-j, 1\right) &\sim \frac{1}{B(j)} \ , && \text{(A.9)}\\
C(j_1, j_2, 0) &= B(j_1)\delta(j_1 - j_2) \ , && \text{(A.10)}\\
C\left(\frac{k}{2}-j_1, \frac{k}{2}-j_2, j_3\right) &\sim B\left(\frac{k}{2}-j_1\right) B\left(\frac{k}{2}-j_2\right) C(j_1, j_2, j_3) \ , && \text{(A.11)}
\end{aligned}$$

where $\sim$ indicates that the identity holds up to a k dependent (j independent) factor.

We start from the following formal expression for the OPE (see [3] for a detailed definition of the OPE in the SL(2,$\mathbb{C}$)/SU(2) WZW model)

$$\Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2) \sim \int dj_i \ Q(j_1, j_2, j_i)\Phi_{j_i}(x_2, z_2) \ , \tag{A.12}$$

where from now on we drop the $x_i, \ z_i$ dependent factors. The coefficient Q can be determined multiplying both sides of eq. (A.12) by Φ_{j_3}, namely

$$\Phi_{j_1}(x_1, z_1)\Phi_{j_2}(x_2, z_2)\Phi_{j_3}(x_3, z_3) \sim \int dj_i \ Q(j_1, j_2, j_i)\Phi_{j_i}(x_2, z_2)\Phi_{j_3}(x_3, z_3) \ , \tag{A.13}$$

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and taking the expectation values as

$$\langle \Phi_{j_1}(x_1,z_1)\Phi_{j_2}(x_2,z_2)\Phi_{j_3}(x_3,z_3)\rangle \sim \int dj\ Q(j_1,j_2,j)\langle \Phi_j(x_2,z_2)\Phi_{j_3}(x_3,z_3)\rangle\ . \tag{A.14}$$

The two point function $\langle \Phi_j \Phi_{j_3}\rangle$ in the right hand side gives two possible contributions which are proportional to $\delta(j-j_3)$ and $\delta(j+j_3-1)$ (see eq. (2.9)). As discussed in reference [12] they both give the same result for Q, namely

$$Q(j_1,j_2,j_3) = \frac{C(j_1,j_2,j_3)}{B(j_3)}\ . \tag{A.15}$$

Let us now repeat this procedure for the four point function. Starting from eq. (A.13) we multiply both sides by $\Phi_{j_4}(x_4,z_4)$ and take the expectation value. One thus obtains

$$\begin{aligned}\langle \Phi_{j_1}\Phi_{j_2}\Phi_{j_3}\Phi_{j_4}\rangle &\sim \int dj\ Q(j_1,j_2,j)\langle \Phi_j(x_2,z_2)\Phi_{j_3}(x_3,z_3)\Phi_{j_4}(x_4,z_4)\rangle \\ &\sim \int dj\ C(j_1,j_2,j)\frac{1}{B(j)}C(j,j_3,j_4)\ .\end{aligned} \tag{A.16}$$

Suppose one of the fields is a spectral flow operator, for instance $j_2 = \frac{k}{2}$. The properties (A.7) and (A.8) allow to perform the integral over j and obtain the coefficient $B(j_1)C(\frac{k}{2}-j_1,j_3,j_4)$ for the three point function $\langle \Phi^{w=1,j_1}_{J,\bar{J}}\Phi_{j_3}\Phi_{j_4}\rangle$ [1].

Similarly, if one starts from a five point function, the OPE (A.12) can be used twice, say for $\Phi_{j_1}\Phi_{j_2}$ and $\Phi_{j_3}\Phi_{j_4}$, with the result

$$\langle \Phi_{j_1}\Phi_{j_2}\Phi_{j_3}\Phi_{j_4}\Phi_{j_5}\rangle \sim \int dj \int dj'\ Q(j_1,j_2,j)Q(j_3,j_4,j')\langle \Phi_j(x_2,z_2)\Phi_{j'}(x_4,z_4)\Phi_{j_5}(x_5,z_5)\rangle\ . \tag{A.17}$$

Again, if there are spectral flow operators, the result simplifies. For instance consider $j_2 = j_4 = \frac{k}{2}$. In this case one can perform the double integral above and obtain

$$\langle \Phi_{j_1}\Phi_{\frac{k}{2}}\Phi_{j_3}\Phi_{\frac{k}{2}}\Phi_{j_5}\rangle \sim B(j_1)B(j_3)C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_3,j_5\right)\ , \tag{A.18}$$

for the coefficient of the three point function $\langle \Phi^{w=1,j_1}_{J_1}\Phi^{w=1,j_3}_{J_3}\Phi_{j_5}\rangle$.

In the more complicated case where we have only one $\Phi_{\frac{k}{2}}$, say $j_4 = \frac{k}{2}$, the double integral turns into a single one of the form

$$\int dj\ \frac{B(j_3)}{B(j)}\ C(j_1,j_2,j)\ C\left(j,\frac{k}{2}-j_3,j_5\right)\ , \tag{A.19}$$

which corresponds to the four point function $\left\langle \Phi_{j_1}\Phi_{j_2}\Phi^{w=1,j_3}_{J}\Phi_{j_5}\right\rangle$.

Finally if one starts from the six point function containing three spectral flow operators and wants to obtain the coefficient for $\langle \Phi^{w=1,j_1}_{J_1}\Phi^{w=1,j_3}_{J_3}\Phi^{w=1,j_6}_{J_6}\rangle$, the OPE can be used three times and the properties (A.7) and (A.8) determine the following corresponding coefficient

$$\langle \Phi_{j_1}\Phi_{\frac{k}{2}}\Phi_{j_3}\Phi_{\frac{k}{2}}\Phi_{\frac{k}{2}}\Phi_{j_6}\rangle \sim B(j_1)B(j_3)B(j_6)C\left(\frac{k}{2}-j_1,\frac{k}{2}-j_3,\frac{k}{2}-j_6\right)\ . \tag{A.20}$$

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B. Useful formulae

In this appendix we collect some useful formulae that have been used in the main body of the article.

The following integral was found in [10] and it was used in section 3 in order to compute the four point function $A_4^{w=1}$ involving one $w = 1$ state, one spectral flow operator and two generic unflowed states

$$\begin{aligned}\int d^2t\, t^p\bar{t}^{\bar{p}}\,|at+b|^{2q}\,|ct+d|^{2r} &= 2i(-1)^{p+\bar{p}}\pi\frac{\Gamma(p+1)\Gamma(q+1)\Gamma(-\bar{p}-q-1)}{\Gamma(-\bar{p})\Gamma(-q)\Gamma(p+q+2)}|d|^{2r}\\ &\times\frac{b^{p+q+1}\bar{b}^{\bar{p}+q+1}}{a^{p+1}\bar{a}^{\bar{p}+1}}\left[{}_2F_1\left(-r,1+p,2+p+q;\frac{cb}{ad}\right){}_2F_1\left(-r,1+\bar{p},2+\bar{p}+q;\frac{\bar{c}\bar{b}}{\bar{a}\bar{d}}\right)\right.\\ &+\lambda\left(\frac{cb}{ad}\right)^{-1-p-q}\left(\frac{\bar{c}\bar{b}}{\bar{a}\bar{d}}\right)^{-1-\bar{p}-q}{}_2F_1\left(-q,-1-p-q-r,-p-q;\frac{cb}{ad}\right)\\ &\times\left.{}_2F_1\left(-q,-1-\bar{p}-q-r,-\bar{p}-q;\frac{\bar{c}\bar{b}}{\bar{a}\bar{d}}\right)\right]\,, \end{aligned} \tag{B.1}$$

where

$$\lambda = \frac{\Gamma(p+q+2)\Gamma(-q-\bar{p}-r-1)\gamma(-q)\Gamma(-\bar{p})\Gamma(p+q+1)}{\Gamma(-q-\bar{p})\Gamma(-q-\bar{p}-1)\gamma(-r)\Gamma(p+1)\Gamma(p+q+r+2)}\,. \tag{B.2}$$

We now write the general result for the integral used in section 3 which was computed in reference [11] (see also [12] for various equivalent expressions), namely

$$\begin{aligned} I &= \int d^2u d^2v u^{\alpha}(1-u)^{\beta}\bar{u}^{\bar{\alpha}}(1-\bar{u})^{\bar{\beta}}v^{\alpha'}(1-v)^{\beta'}\bar{v}^{\bar{\alpha}'}(1-\bar{v})^{\bar{\beta}'}|u-v|^{4\sigma}\\ &= -\frac{1}{4}\left(C^{12}[\alpha_i,\alpha_i']P^{12}[\bar{\alpha}_i,\bar{\alpha}_i'] + C^{21}[\alpha_i,\alpha_i']P^{21}[\bar{\alpha}_i,\bar{\alpha}_i']\right)\,, \end{aligned} \tag{B.3}$$

where

$$\begin{aligned} C^{ab}[\alpha_i,\alpha_i'] &= \frac{\Gamma(1+\alpha_a+\alpha_a'-k')\Gamma(1+\alpha_b+\alpha_b'-k')}{\Gamma(k'-\alpha_c-\alpha_c')}\\ &\quad\times G\begin{bmatrix}\alpha_a'+1,\alpha_b+1,k'-\alpha_c-\alpha_c'\\ \alpha_a'-\alpha_c+1,\alpha_b-\alpha_c'+1\end{bmatrix}\,. \end{aligned} \tag{B.4}$$

Here

$$G\begin{bmatrix}a,b,c\\ e,f\end{bmatrix} = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(e)\Gamma(f)}{}_3F_2(a,b,c;e,f;1)\,, \tag{B.5}$$

$$\begin{aligned} &\alpha_1=\alpha, \qquad \alpha_2=\beta, \qquad \alpha_3=\gamma, \qquad \alpha+\beta+\gamma+1=k'=-2\sigma-1,\\ &\alpha_1'=\alpha', \qquad \alpha_2'=\beta', \qquad \alpha_3'=\gamma', \qquad \alpha'+\beta'+\gamma'+1=k'=-2\sigma-1, \end{aligned} \tag{B.6}$$

and similarly for $\bar{\alpha}_i$ and $\bar{\alpha}_i'$. P^{12} and P^{21} are given by

$$\begin{bmatrix}P^{12}\\ P^{21}\end{bmatrix} = A_\beta\begin{bmatrix}C^{23}\\ C^{32}\end{bmatrix} = A_\alpha^T\begin{bmatrix}C^{31}\\ C^{13}\end{bmatrix}\,, \tag{B.7}$$

with

$$A_\beta = -4 \begin{bmatrix} s(\beta)s(\beta') & -s(\beta)s(\beta'-k') \\ -s(\beta')s(\beta-k') & s(\beta)s(\beta') \end{bmatrix}, \tag{B.8}$$

and $s(x) = sin(\pi x)$.

The following identities among ${}_3F_2(a,b,c;e,f;1) = F\left[\begin{smallmatrix} a,b,c \\ e,f \end{smallmatrix}\right]$ or $G\left[\begin{smallmatrix} a,b,c \\ e,f \end{smallmatrix}\right]$ have been used in section 3 to obtain the two point function of spectral flowed states,

$$\begin{aligned} G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} &= \frac{\Gamma(b)\Gamma(c)}{\Gamma(e-a)\Gamma(f-a)} G\begin{bmatrix} e-a, f-a, u \\ u+b, u+c \end{bmatrix} \\ &= \frac{\Gamma(b)\Gamma(c)\Gamma(u)}{\Gamma(f-a)\Gamma(e-b)\Gamma(e-c)} G\begin{bmatrix} a, e-b, e-c \\ e, u+a \end{bmatrix}, \\ G\begin{bmatrix} a,b,c \\ e,f \end{bmatrix} &= \frac{s(e-b)s(f-b)}{s(a)s(c-b)} G\begin{bmatrix} b, 1+b-e, 1+b-f \\ 1+b-c, 1+b-a \end{bmatrix} \\ &\quad + \frac{s(e-c)s(f-c)}{s(a)s(b-c)} G\begin{bmatrix} c, 1+c-e, 1+c-f \\ 1+c-b, 1+c-a \end{bmatrix}, \end{aligned} \tag{B.9}$$

where $u = e+f-a-b-c$.

C. Three point function including three $w = 1$ fields

In this appendix we aim at computing a three point function involving three $w = 1$ fields following the procedure introduced in section 5. Namely we want to compute

$$\left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\right\rangle, \tag{C.1}$$

which should be obtained from the six point function including three spectral flow operators

$$\langle \Phi_{j_1}(y_1,\zeta_1)\Phi_{\frac{k}{2}}(x_1,z_1)\Phi_{j_2}(y_2,\zeta_2)\Phi_{\frac{k}{2}}(x_2,z_2)\Phi_{\frac{k}{2}}(x_3,z_3)\Phi_{j_3}(y_3,\zeta_3)\rangle . \tag{C.2}$$

The fusion of the spectral flow operators with the remaining fields is expected to give rise to the $w = 1$ fields in (C.1). Here we propose an approach which is alternative to the exhaustive one in section 3, and will allow us to compute (C.1) for specific relations among the spins of the fields. In fact, this will be done for the following cases

$$i) \qquad j_1 + j_2 + j_3 = \frac{k}{2}, \tag{C.3}$$

$$ii) \qquad j_1 + j_2 - j_3 = \frac{k-2}{2} \quad \text{and permutations}, \tag{C.4}$$

and

$$iii) \qquad j_1 + j_2 - j_3 = \frac{4-k}{2} \quad \text{and permutations}. \tag{C.5}$$

The procedure exemplified here may be useful in order to compute particular expressions for correlators involving many units of spectral flow, where exhaustive calculations imply increasing difficulties.

Our strategy is as follows. We will first perform an intermediate step in which we will spectral flow only two operators in (C.2) in order to obtain the modified KZ and null vector equations to be satisfied by the following four point function (after redefining $(y_3, \zeta_3) \longrightarrow (x_4, z_4)$)

$$\mathcal{A}_4^w \equiv \left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi_{\frac{k}{2}}(x_3,z_3)\Phi_{j_3}(x_4,z_4)\right\rangle \ . \tag{C.6}$$

Then we will propose an appropriate *ansatz* for the solution, and finally spectral flow one last time in order to find (C.1).

So, by performing calculations analogous to those in the previous sections, we find that the above four point function obeys the following modified KZ equation which arises from the spectral flow operator inserted at (x_3, z_3) in (C.2)

$$\begin{aligned}\left(j_1 - J_1 + \frac{k}{2} - 1\right)\frac{x_{31}}{z_{31}^2}\,\mathcal{A}_4^w(J_1+1,J_2) + \left(j_2 - J_2 + \frac{k}{2} - 1\right)\frac{x_{32}}{z_{32}^2}\,\mathcal{A}_4^w(J_1,J_2+1)\\ = \left[-\frac{\partial}{\partial z_3} + \frac{1}{z_{31}}\left(x_{31}\frac{\partial}{\partial x_1} - J_1\right) + \frac{1}{z_{32}}\left(x_{32}\frac{\partial}{\partial x_2} - J_2\right)\right.\\ \left. + \frac{1}{z_{43}}\left(x_{43}\frac{\partial}{\partial x_4} + j_3\right)\right]\mathcal{A}_4^w(J_1,J_2)\ .\end{aligned} \tag{C.7}$$

In addition, due to the three spectral flow operators in (C.2) we accordingly get three modified null vector equations for $\mathcal{A}_4^w$. They read

$$\begin{aligned}\left(j_1 - J_1 + \frac{k}{2} - 1\right)\frac{x_{31}^2}{z_{31}^2}\,\mathcal{A}_4^w(J_1+1,J_2) + \left(j_2 - J_2 + \frac{k}{2} - 1\right)\frac{x_{32}^2}{z_{32}^2}\,\mathcal{A}_4^w(J_1,J_2+1)\\ = \left[\frac{x_{31}}{z_{31}}\left(x_{31}\frac{\partial}{\partial x_1} - 2J_1\right) + \frac{x_{32}}{z_{32}}\left(x_{32}\frac{\partial}{\partial x_2} - 2J_2\right) - \frac{x_{43}}{z_{43}}\left(x_{43}\frac{\partial}{\partial x_4} + 2j_3\right)\right]\mathcal{A}_4^w(J_1,J_2)\ ,\end{aligned} \tag{C.8}$$

$$\begin{aligned}-\left(j_1 + J_1 - \frac{k}{2} - 1\right)\mathcal{A}_4^w(J_1-1,J_2) - \left(j_2 - J_2 + \frac{k}{2} - 1\right)\frac{x_{21}^2}{z_{21}^2}\,\mathcal{A}_4^w(J_1,J_2+1)\\ = \left[\frac{x_{21}}{z_{21}}\left(x_{21}\frac{\partial}{\partial x_2} + 2J_2\right) + \frac{x_{31}}{z_{31}}\left(x_{31}\frac{\partial}{\partial x_3} + k\right) + \frac{x_{41}}{z_{41}}\left(x_{41}\frac{\partial}{\partial x_4} + 2j_3\right)\right]\mathcal{A}_4^w(J_1,J_2)\ ,\end{aligned} \tag{C.9}$$

and

$$\begin{aligned}\left(j_1 - J_1 + \frac{k}{2} - 1\right)\frac{x_{21}^2}{z_{21}^2}\,\mathcal{A}_4^w(J_1+1,J_2) + \left(j_2 + J_2 - \frac{k}{2} - 1\right)\mathcal{A}_4^w(J_1,J_2-1)\\ = \left[\frac{x_{21}}{z_{21}}\left(x_{21}\frac{\partial}{\partial x_1} - 2J_1\right) - \frac{x_{32}}{z_{32}}\left(x_{32}\frac{\partial}{\partial x_3} + k\right) - \frac{x_{42}}{z_{42}}\left(x_{42}\frac{\partial}{\partial x_4} + 2j_3\right)\right]\mathcal{A}_4^w(J_1,J_2)\ .\end{aligned} \tag{C.10}$$

From the results of section 4, we expect that $\mathcal{A}_4^w$ had the same functional form as an unflowed four point function, but with appropriate modified expressions for the spins and conformal dimensions of the $w = 1$ fields. So we look for a solution of the following form

$$\begin{aligned}\mathcal{A}_4^w &= \mathcal{D}_1(j_1,J_1,j_2,J_2,j_3)\,\mathcal{D}_2(j_1,\bar{J}_1,j_2,\bar{J}_2,j_3)\,\mathcal{F}(z,x)\,\bar{\mathcal{F}}(\bar{z},\bar{x})\\ &\times\left(x_{43}^{J_1+J_2-k/2-j_3}x_{42}^{-2J_2}x_{41}^{J_2+k/2-J_1-j_3}x_{31}^{j_3-J_1-J_2-k/2}\right)\\ &\times\left(z_{43}^{\Delta_1^{w=1}+\Delta_2^{w=1}+k/4-\Delta_3}z_{42}^{-2\Delta_2^{w=1}}z_{41}^{\Delta_2^{w=1}-k/4-\Delta_1^{w=1}-\Delta_3}z_{31}^{\Delta_3-\Delta_1^{w=1}-\Delta_2^{w=1}+k/4}\right)\\ &\times\text{(antiholomorphic part)}\ .\end{aligned} \tag{C.11}$$

Here we are assuming that there is only one state in each intermediate channel, similarly as in the construction of [1]. We have shown in appendix A how the spins of the intermediate states are fixed due to the presence of the spectral flow operators, thus avoiding the integral introduced in [3]. Notice that $\mathcal{D}_1$ and $\mathcal{D}_2$ are the coefficients of the four point function depending on the right and left spins of the string states. The cross ratios z, x are as in (5.4).

Now we make the following *ansatz* for the functions $\mathcal{F}$, $\bar{\mathcal{F}}$

$$\mathcal{F} = z^{\alpha}(1-z)^{\beta}x^{\mu}(1-x)^{\nu}(z-x)^{\rho} , \qquad \bar{\mathcal{F}} = \bar{z}^{\bar{\alpha}}(1-\bar{z})^{\bar{\beta}}\bar{x}^{\bar{\mu}}(1-\bar{x})^{\bar{\nu}}(\bar{z}-\bar{x})^{\bar{\rho}} , \tag{C.12}$$

where the factors of $z, 1-z, x$ and $1-x$ are suggested by the standard structure of singularities in conformal field theory and string theory, namely those appearing at the boundary of the moduli space when two or more vertex operator insertions collide on the worldsheet. The dependence on $z-x$ was found in reference [1] where it was shown to be required by monodromy invariance of the four point amplitude when the holomorphic and antiholomorphic parts are combined. The singularity at $z = x$ was interpreted there as due to instanton effects.

In principle, no further poles arise in presence of spectral flowed states, as noticed *e.g.* from direct inspection of (5.5). In fact, plugging (C.11) and (C.12) into (C.7)–(C.10) we find that the *ansatz* (C.12) is the solution for relations (C.3)–(C.5) among the spins. Calculations involve a last step in which we spectral flow the field Φ_{j_3} in (C.6) in order to get the three point function (C.1). This is done using the prescription (2.5) and the integration procedure is similar as in section 3. Since instead of computing (C.6) we could also find the three point function (C.1) starting from correlators

$$\mathcal{A}'^{w}_{4} \equiv \left\langle \Phi_{j_1}(y,\zeta)\Phi_{\frac{k}{2}}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\right\rangle , \tag{C.13}$$

or

$$\mathcal{A}''^{w}_{4} \equiv \left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi_{j_2}(y,\zeta)\Phi_{\frac{k}{2}}(x_2,z_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\right\rangle , \tag{C.14}$$

we must also require that the final result does not depend on the intermediate path we follow to compute it, which imposes further restrictions.

Following all the prescriptions above, using also the results in appendix A and after performing some algebra, we arrive at the following three point functions for spins related as in (C.3)–(C.5)

$$\begin{aligned}
&\left\langle \Phi^{w=1,j_1}_{J_1,\bar{J}_1}(x_1,z_1)\Phi^{w=1,j_2}_{J_2,\bar{J}_2}(x_2,z_2)\Phi^{w=1,j_3}_{J_3,\bar{J}_3}(x_3,z_3)\right\rangle \\
&\quad \sim B(j_1)B(j_2)B(j_3)C\left(\frac{k}{2}-j_1, \frac{k}{2}-j_2, \frac{k}{2}-j_3\right) \\
&\qquad \times\pi\,\gamma(1-2j_1)\,\frac{\Gamma(j_1+J_1-k/2)}{\Gamma(1-j_1+J_1-k/2)}\,\frac{\Gamma(j_1-\bar{J}_1+k/2)}{\Gamma(1-j_1-\bar{J}_1+k/2)} \\
&\qquad \times\pi\,\gamma(1-2j_2)\,\frac{\Gamma(j_2+J_2-k/2)}{\Gamma(1-j_2+J_2-k/2)}\,\frac{\Gamma(j_2-\bar{J}_2+k/2)}{\Gamma(1-j_2-\bar{J}_2+k/2)} \\
&\qquad \times\pi\,\gamma(1-2j_3)\,\frac{\Gamma(j_3+J_3-k/2)}{\Gamma(1-j_3+J_3-k/2)}\,\frac{\Gamma(j_3-\bar{J}_3+k/2)}{\Gamma(1-j_3-\bar{J}_3+k/2)}
\end{aligned}$$

$$\times \left(x_{32}^{J_1-J_2-J_3} x_{31}^{J_2-J_1-J_3} x_{21}^{J_3-J_1-J_2} \right)$$
$$\times \left(z_{32}^{\Delta_1^{w=1}-\Delta_2^{w=1}-\Delta_3^{w=1}} z_{31}^{\Delta_2^{w=1}-\Delta_1^{w=1}-\Delta_3^{w=1}} z_{21}^{\Delta_3^{w=1}-\Delta_1^{w=1}-\Delta_2^{w=1}} \right)$$
$$\times (\text{antiholomorphic part}) \,, \tag{C.15}$$

up to some k dependent coefficient.

Explicit results for other relations among the spins could in principle be obtained using a more involved *ansatz* than the one in (C.12), possibly involving an hypergeometric function.

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